The Riemann-Stieltjes Integral

The Riemann integral solves the problems of defining the area under the graph of a continuous function f, defined on a closed interval $[a\ b]$

Partition of an Interval # ✓

Let $[a, b]$ be a closed interval. Any finite ordered set

$$P = \{a = x_0 < x_1 < x_2 \dots x_n = b\}$$

which divides the interval $[a, b]$ into n-subintervals $[x_0\ x_1]$ $[x_1, x_2]$ ------ $[x_{i-1}, x_i]$ ---- $[x_{n-1}, x_n]$ is called a partition of the interval $[a\ b]$

OR

A partition P of a closed interval $[a, b]$ is a finite set $\{x_0 = a, x_1, x_2 \dots, x_n = b\}$ of real numbers such that

$$a = x_0 < x_1 < x_2 < x_3 \dots < x_n = b$$

The closed intervals $[x_{i-1}, x_i]$ for $i = 1, 2, \dots n$ are called Component intervals of the partition P.

Notations

We shall use a single symbol Δx_i to denote the length of the interval $[x_{i-1}, x_i]$. The symbols M_i & m_i are used to denote the supremum and infimum of f on $[x_{i-1}, x_i]$. Upper and lower bounds of f on $[a, b]$ will be denoted by M, m respectively. The family of all possible partitions of $[a\ b]$ will be denoted by $\mathcal{P}[a\ b]$. Note that

$$\sum_P \Delta x_i = x_1 - a + x_2 - x_1 + x_3 - x_2 + \dots + x_n = b - x_{n-1} = b - a$$

Norm of a Partition

Let $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$ be any partition of $[a, b]$. The length of the greatest of all the component intervals $[x_{i-1}, x_i]$ $i = 1, 2 \dots n$ is called the norm of partition P and is denoted by $\|P\|$.

Thus

$$\|P\| = Max\{(x_i - x_{i-1}) : 1 \leq i \leq n\}$$

$$= Max\{\Delta x_i : 1 \leq i \leq n\}$$

Norm of a partition is also called the mesh of the partition.

Refinement of a Partition

Let P_1 & P_2 be any two partitions of $[a, b]$ such that

$$P_1 \subset P_2$$

then P_2 is a refinement of P_1 or P_2 is said to be finer than P_1

Common Refinement

If P_1, P_2, P^* are any three partitions of $[a, b]$ and $P^* = P_1 \cup P_2$, then P^* is called common refinement of P_1 & P_2

Upper and Lower Darboux Sums

Let f be a bounded real function defined on $[a, b]$. Corresponding to each partition P of $[a, b]$ put

$$M_i = Sup\, f(x) \qquad x_{i-1} \leq x \leq x_i$$

$$m_i = Inf\, f(x) \qquad x_{i-1} \leq x \leq x_i$$

Then the sum $U(P, f)$, $L(P, f)$ defined by

$$U(P,f) = M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n = \sum_{i=1}^{n} M_i\Delta x_i$$

$$L(P,f) = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n = \sum_{i=1}^{n} m_i\Delta x_i$$

are respectively called upper and Lower Darboux Sums, Corresponding to the partition P of $[a,b]$

Note# Note that $U(P,-f) = -L(P,f)$

and

$$L(P,-f) = -U(P,f)$$

Explanation#

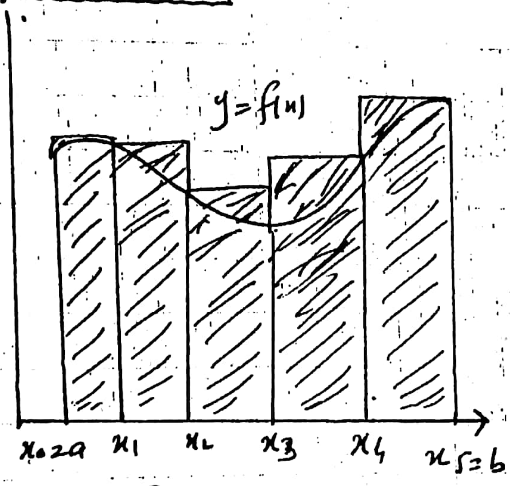


fig ②

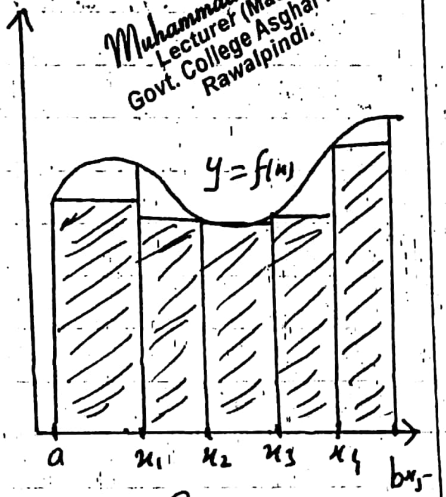


fig ①

Consider a simple partition $P = \{a = x_0, x_1, x_2, x_3, x_4, x_5 = b\}$ and suppose that f has graph as shown in fig above. Then

(a) $L(P,f)$ = total shadded area in fig ① i.e it is the total area of inscribed rectangles of the curve $y = f(x)$ from $x = a$ to $x = b$. It approximate the area under f from below

(b) $U(P,f)$ = total shadded area in fig ② i.e it is the total area of Circumscribed rectangles of the curve $y = f(x)$ from $x = a$ to $x = b$ and it approximate the area under f from above.

(C) If A is the exact area between the graph

of f, then we note that

$$L(P,f) \leq A \leq U(P,f)$$

Note # Whe the norm (mesh) of Partition P is decreased by increasing the no of points of division (intermediate points), lower sum increases while the upper sum decreases. Also

$$L(P,f) \leq U(P,f)$$

Properties of Darboux Sums

1) # Let f be a bounded function on [a b] and P be any partition of [a, b], then U(P, f) and L(P, f) are bounded and

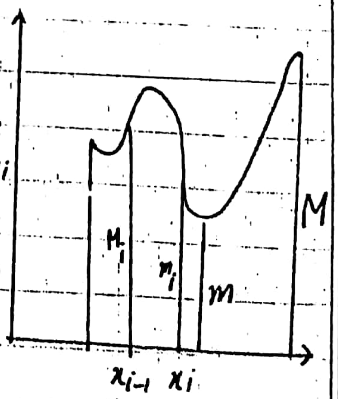
$$U(P,f) \geq L(P,f)$$

OR

$$U(P,f) \geq L(P,f) \quad \forall \text{ partitions of } [a,b]$$

Proof # Let M, m be the upper and lower bounds of f on [a b]. Then by definition

$$m \leq m_i \leq M_i \leq M$$



$$\Rightarrow \sum_{i=1}^{n} m\Delta x_i \leq \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i \leq \sum_{i=1}^{n} M \Delta x_i$$

$$m \sum_{i=1}^{n} \Delta x_i \leq L(P,f) \leq U(P,f) \leq M \sum \Delta x_i$$

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

Thus $U(P,f) \geq L(P,f)$ and $U(P,f)$, $L(P,f)$ are bounded for all partitions.

2) # If P_1, P_2 are any two partitions of $[a, b]$ and $P_1 \subset P_2$, then

(a) $U(P_2, f) \leq U(P_1, f)$

i.e the upper Darboux sum cannot increase by refineness of the partition

(b) $L(P_2, f) \geq L(P_1, f)$ i.e the Lower Darboux sum can not decrease with refiness of the Partition.

Proof # Let $P_1 = \{x_0 = a < x_1 < x_2 - - - - x_n = b\}$ and P_2 be a partition with k additional points $c_1, c_2 - - - - c_k$ than those of P_1.

Let $P_2' = \{c_1\} \cup P_1$ where $x_{r-1} < c_1 < x_r$ and M_{r_1}, M_{r_2} be the upper bounds of f on $[x_{r-1}, c_1]$ and $[c_1, x_r]$ respectively. Then

$$\max\{M_{r_1}, M_{r_2}\} \leq M_r \quad \text{where } M_r \text{ is bound of } f \text{ on } [x_{r-1}, x_r]$$

and

$$U(P_1) - U(P_2') = \sum_{i=1}^{n} M_i \Delta x_i - \Big[\sum_{i=1}^{r-1} M_i \Delta x_i + M_{r_1}(c - x_{r-1})$$
$$+ M_{r_2}(x_r - c_1) + \sum_{i=r+1}^{n} M_i \Delta x_i\Big]$$

$$= \sum_{i=1}^{r-1} M_i \Delta x_i + M_r(x_r - x_{r-1}) + \sum_{i=r+1}^{n} M_i \Delta x_i - \sum_{i=1}^{r-1} M_i \Delta x_i$$
$$- M_{r_1}(c_1 - x_{r-1}) - M_{r_2}(x_r - c_1) - \sum_{i=r+1}^{n} M_i \Delta x_i$$

$$= M_r(x_r - x_{r-1}) - M_{r_1}(c_1 - x_{r-1}) - M_{r_2}(x_r - c_1)$$

$$= M_r(x_r + c_1 - c_1 - x_{r-1}) - M_{r_1}(c_1 - x_{r-1}) - M_{r_2}(x_r - c_1)$$

$$= M_r(c_1 - x_{r-1}) + M_r(x_r - c_1) - M_{r_1}(c_1 - x_{r-1}) - M_{r_2}(x_r - c_1)$$

$$= (M_r - M_{r_1})(c_1 - x_{r-1}) + (M_r - M_{r_2})(x_r - c_1) \geq 0$$

$\because M_k \geq M_{k_1}, M_k \geq M_{k_2}$ $c_1 - x_{k-1} \geq 0, x_r - c_1 \geq 0$

$\therefore U(P_1) \geq U(P_2')$

Repeating this process with $P_2', P_2'' = P_2' \cup \{c_2\}$

$-------, P_2^k = P_2^{k-1} \cup \{c_k\} = P_2$

we get

$U(P_2') \geq U(P_2'') \geq ------ \geq U(P_2^{k-1}) \geq U(P_2^k)$, where $P_2^k = P_2$

Thus $P_1 \subset P_2 \Rightarrow U(P_1) \geq U(P_2)$

Similarly it can be proved that

$P_1 \subset P_2 \Rightarrow L(P_1) \leq L(P_2)$

From this property it follows that

$$P_1 \subset P_2 \Rightarrow U(P_1) - L(P_1) \geq U(P_2) - L(P_2)$$

3) # $L(P_1) \leq U(P_2) \quad \forall P_1, P_2 \in \mathcal{P}[a\ b]$

i.e for any partitions of $[a\ b]$

Proof # Let P be a partition of $[a\ b]$ finer than both P_1 & P_2, we have

$$L(P_1) \leq L(P) \leq U(P) \leq U(P_2)$$

$$\Rightarrow L(P_1) \leq U(P_2)$$

4) # (Estimate of Difference in sums)

If $P_1, P_2 \in \mathcal{P}[a\ b]$, $\|P_1\| \leq \delta$ and $|f(x)| < k \quad \forall x \in [a\ b]$, Then for $P_1 \subset P_2$ with p additional points

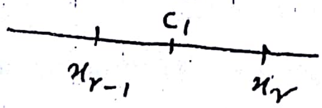
$$U(P_1) \leq U(P_2)\ 2pk\delta$$

$$L(P_2) \leq L(P_1) + 2pk\delta$$

Proof # Let $P_1 = \{x_0 = a, x_1, x_2 ------ x_n = b\}$ and P_2 has additional points $c_1, c_2 ------ c_p$

than those of P_1.

Let $P_2' = \{c_1\} \cup P_1$ where $x_{r-1} < c_1 < x_r$



Let M_{r1}, M_{r2} be upper bounds of f on $[x_{r-1}, c_1]$ and $[c_1, x_r]$ respectively. Then

$$U(P_1) - U(P_2') = (M_r - M_{r_1})(c_1 - x_{r-1}) + (M_r - M_{r_2})(x_r - c) \longrightarrow ①$$

$$\because \quad |f(x)| < k \quad \forall x \in [a, b]$$

$$\Rightarrow -k < f(x) < k \quad \forall x \in [a, b]$$

$$\Rightarrow -k \leq \begin{matrix} M_{r_1} \\ M_{r2} \end{matrix} \leq M_r \leq k$$

$$\Rightarrow 0 \leq \begin{matrix} M_r - M_{r1} \\ M_r - M_{r2} \end{matrix} \leq 2k.$$

Using in ①

$$U(P_1) - U(P_2') \leq 2k(c_1 - x_{r-1}) + 2k(x_r - c)$$
$$= 2k(x_r - x_{r-1})$$
$$= 2k\,\Delta x_r$$
$$\leq 2k\delta \longrightarrow ② \quad \because \|P_1\| \leq \delta$$

Thus

$$U(P_1) \leq U(P_2') + 2k\delta$$

on repeating this process with $P_2', P_2'', \ldots, P_2^{p-1}, P_2^{p}$ we get

$$U(P_2') \leq U(P_2'') + 2k\delta \quad \forall \|P_1\| \leq \delta$$
$$U(P_2'') \leq U(P_2''') + 2k\delta$$
$$\cdots\cdots\cdots$$
$$U(P_2^{p-1}) \leq U(P_2^{p}) + 2k\delta \quad \forall \|P_1\| \leq \delta$$

By using all these inequalities, we have

$$U(P_1) \le U(P_2) + 2pk\delta$$

Similarly, $L(P_2) \le L(P_1) + 2pk\delta$ for $P_1 \subset P_2$ with p additional points.

5) # If P is a partition of $[a\ b]$ and f, g are bounded on $[a, b]$, then

(a) $U(P, f+g) \le U(P, f) + U(P, g)$

(b) $L(P, f+g) \geq L(P, f) + L(P, g)$.

Proof # Let Δx_r be the length of rth sub-interval of P and M_r', m_r' ; M_r'', m_r'' be the respective bounds of f & g on $[x_{r-1}\ x_r]$. Then

$$f(x) \le M_r' \qquad g(x) \le M_r'' \qquad \forall x \in [x_{r-1}, x_r]$$

$$\Rightarrow f(x) + g(x) \le M_r' + M_r'' \qquad \forall x \in [x_{r-1}\ x_r]$$

$$\Rightarrow M_r \le M_r' + M_r'' \qquad \forall x \in [x_{r-1}, x_r]$$

where M_r is upper bound of $f+g$ on $[x_{r-1}\ x_r]$

Thus $\sum_{r=1}^{n} M_r \Delta x_r \le \sum_{r=1}^{n} M_r' \Delta x_r + \sum M_r'' \Delta x_r$

$$\Rightarrow U(P, f+g) \le U(P, f) + U(P, g)$$

Similarly

$$f \geq m' \quad g \geq m'' \qquad \text{on } \Delta x_r$$

$$\Rightarrow f+g \geq m' + m'' \qquad \text{on } \Delta x_r$$

$$\Rightarrow m_r \geq m' + m'' \qquad \text{on } \Delta x_r$$

where m_r is the lower bound of $f+g$ on Δx_r

Thus

$$\sum m_r \Delta x_r \geq \sum m_r' \Delta x_r + \sum m_r'' \Delta x_r$$

$$\Rightarrow L(P, f+g) \geq L(P, f) + L(P, g)$$

6)# If P_1, P_2 are partitions of $[a\ b]$, then
$L(P_1, f) \leq U(P_2, f)$

Proof# Let $T = P_1 \cup P_2$, then T is a common refinement of both P_1 & P_2. So

$$L(P_1, f) \leq L(T, f) \leq U(T, f) \leq U(P_2, f)$$

$$\Rightarrow L(P_1, f) \leq U(P_2, f)$$

Remarks# If $\{P_n\}$ is a sequence of partitions of $[a\ b]$ such that $P_n \subset P_{n+1}$ $\forall n \geq 1$, then the sequences $\{S(P_n, f)\} = \{U(P_n, f)\}$

and $\{L(P_n, f)\}$ of upper and lower sums are monotonically decreasing and increasing. Since according to property ①

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

for all partitions P of $[a\ b]$
, Therefore these sequences are also bounded. These sequences are convergent because bounded monotone sequences are always convergent

Oscillatory Sum#

We have

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta x_i$$

$$= \sum_{i=1}^{n} (M_i - m_i) \Delta x_i = \sum_{i=1}^{n} O_i \Delta x_i$$

where O_i denotes the oscillation of the function in the sub-interval $[x_{i-1}, x_i]$. The sum $\sum_{i=1}^{n} O_i \Delta x_i$ which is called the oscillatory sum is denoted by $W(P)$.

As the oscillation O_i can not be negative, it follows that each oscillatory sum consists of the sum of a finite number of non-negative terms.

Upper and lower Integrals#

If f is a function bounded on $[a, b]$, then for every partition P of $[a, b]$, $U(P, f)$ & $L(P, f)$ exist and the sets of upper and lower sums are bounded. So that $\text{Inf}\, U(P, f)$ & $\text{Sup}\, L(P, f)$ exist uniquely as $\|P\| \to 0$. This leads to the following definition of the upper and lower integrals on $[a\ b]$

Upper Integral

If f is bounded on $[a, b]$, the upper Riemann integral of f on $[a, b]$, denoted by $\overline{\int}_a^b f(x)dx$ or $\int_a^{\bar{b}} f(x)dx$ is the infimum of the set of the upper sums. i.e

$$\int_a^{\bar{b}} f(x)\,dx = \int_a^{\bar{b}} f\,dx = \text{Inf}\{U(P, f) : P \in \mathcal{P}[a\ b]\}$$

$$= \text{Inf}\, U(P, f)$$

where infimum is taken over all the partition i.e when $\|P\| \to 0$

Lower Integral

If f is bounded on $[a, b]$, then the lower Riemann integral denoted by

$\underline{\int_a^b} f(x)\,dx$ or $\int_{\underline{a}}^{b} f(x)\,dx$ is the Supremum of the set of all lower sums i.e

$$\underline{\int_a^b} f(x)\,dx = \operatorname{Sup}\{L(P,f) : P \in \mathcal{P}[a,b]\}$$

$$= \operatorname{Sup} L(P)$$, where the supremum is taken over all the partitions.

Riemann Integrable Function#

A function f bounded on $[a,b]$ is said to be Riemann integrable or simply integrable over $[a,b]$, if its upper and lower integrals are equal. The common value of these integrals is called the Riemann integral or simply the integral and is denoted by

$$\int_a^b f(x)\,dx$$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

Note# 1)# The numbers a, b respectively are called lower and the upper limits of integration.

2)# The definition of integrability as given above is based on the notion of bounds. Another approach based on the notion of of limits is given later on.

3)# It should be clearly understood that not every bounded function is integrable i.e there do exist bounded function f for which

$$\int_a^{\overline{b}} f\,dx \neq \underline{\int_a^b} f(x)\,dx$$

4)# The statement $\int_a^b f(x)\,dx$ exists implies that the function f is bounded and integrable over $[a\ b]$

5)# The concept of integrability of a function over an interval as introduced here has following

two important limitations.

(i) the function is bounded.

(ii) the interval is finite

so that neither of the end points of the interval is infinite.

In improper integral we shall see how these limitations can be removed and the concept can be generalised so as to be applicable sometimes even to cases where the function is not bounded or where one or both the limits of integration are infinite.

If f is a Riemann integrable over $[a\ b]$ we write it as

$f \in R[a\ b]$, where $R[a\ b]$ denotes the set of all Riemann integrable functions over $[a\ b]$.

Example # If f is defined on $[0\ 1]$ by

$$f(x) = x \quad \forall x \in [0\ 1], \text{ then}$$

prove that $f \in R[0\ 1]$ & $\int_0^1 f(x)dx = \frac{1}{2}$.

Sol # Here we make a partition n sub-intervals each of equal length given by

$$\frac{1-0}{n} = \frac{1}{n}$$

Then partition is

$$P = \{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots\cdots, \frac{n}{n} = 1\}$$

Let the sub-intervals be

$$I_r = \left[\frac{r-1}{n}\ \frac{r}{n}\right] \quad \text{for } r = 1, 2, \cdots n$$

If M_r, m_r be respectively the supremum and infimum of the function f in I_r. Then

$$M_r = f(\tfrac{r}{n}) = \frac{r}{n}$$

$$m_r = f(\tfrac{r-1}{n}) = \frac{r-1}{n}$$

$$U(P,f) = \sum_{r=1}^{n} M_r \Delta x_r$$

$$= \sum_{r=1}^{n} \frac{r}{n} \cdot \frac{1}{n} \qquad \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

$$= \frac{1}{n^2} \sum_{r=1}^{n} r = \frac{1}{n^2}\left[\frac{1}{2} n(n+1)\right]$$

$$= \frac{n+1}{2n} = \frac{1}{2}\left(1+\frac{1}{n}\right)$$

$$\int_0^{1} f(x)\,dx = \operatorname*{Inf}_{\|P\| \to 0} U(P,f)$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2} \to (1)$$

$$L(P,f) = \sum_{r=1}^{n} m_r \Delta x_r$$

$$= \sum_{r=1}^{n} \left(\frac{r-1}{n}\right) \cdot \frac{1}{n}$$

$$= \frac{1}{n^2} \sum_{r=1}^{n} (r-1)$$

$$= \frac{1}{n^2}\left[\frac{1}{2}(n-1)(n-1+1)\right] = \frac{1}{2}\left(\frac{n-1}{n}\right)$$

$$= \frac{1}{2}\left(1-\frac{1}{n}\right) \to (*)$$

$$\Rightarrow \int_0^1 f(x)\,dx = \operatorname*{Sup}_{\substack{\|P\|\to 0 \\ n\to\infty}} L(P,f)$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1-\frac{1}{n}\right) = \frac{1}{2} \to (2)$$

By (1) and (2)

$$\int_0^1 f(x)\,dx = \int_0^1 f\,dx = \frac{1}{2}$$

$$\Rightarrow f \in R[0, 1] \text{ and } \int_0^1 f(x)\,dx = \frac{1}{2}$$

Example # Show that a constant function is Riemann integrable.

Sol # Let a function f be defined on $[a\ b]$ by

$$f(x) = c \quad \forall x \in [a\ b]$$

where c is a constant.

Let any partition of $[a\ b]$ be

$$P = \{a = x_0, x_1, x_2 - - - x_r - - - x_n = b\}$$

Let its sub-intervals be

$$\cancel{\Delta x_r} = [x_{r-1}, x_r] \quad r = 1, 2, - - - -$$

Let Δx_r be the length of this interval, then

$$\Delta x_r = x_r - x_{r-1}$$

Let M_r, m_r be respectively the supremum and infimum of the function f in $I_r = [x_{r-1}, x_r]$.

Then

$$M_r = c \qquad m_r = c \quad \text{as } f(x) = c \ \forall x \in [a\ b]$$

$$U(P,f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} c\,(x_r - x_{r-1})$$

$$= c[(x_1 - x_0) + (x_2 - x_1) + (x_3 - x_2) + - - - (x_n - x_{n-1})]$$

$$= c(x_n - x_0) = c(b-a) = \text{Constant}$$

$$L(P,f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} c\,\Delta x_r$$

$$= c(b-a) = \text{Constant}$$

$$\overline{\int_a^b} f(x)\,dx = \underset{\substack{\|P\| \to 0 \\ n \to \infty}}{\text{Inf}}\ U(P,f) = c(b-a)$$

$$\int_a^b f(x)\,dx = \sup_{\substack{\|P\|\to 0 \\ n\to\infty}} L(P,f)$$

$$= c(b-a)$$

Hence $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = c(b-a)$

and so $f \in R[a\ b]$ and

$$\int_a^b f(x)\,dx = c(b-a)$$

Example

If $f(x)$ be defined on $[0\ 1]$ as follows

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$$

Then prove that f is not Riemann integrable over $[0\ 1]$

Sol # Let any partition of $[0\ 1]$ be

$$P = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$$

with sub-intervals

$$I_r = [x_{r-1}, x_r] \quad r = 1, 2, \ldots n$$

clearly

$$M_r = 1 \qquad m_r = -1$$

$$U(P,f) = \sum_{r=1}^{n} M_r \Delta x_r$$

$$= \sum_{r=1}^{n} 1\cdot(x_r - x_{r-1})$$

$$= x_n - x_0 = 1 - 0 = 1$$

$$L(P,f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum -1(x_r - x_{r-1})$$

$$= -(x_n - x_0) = -1$$

$$\int_0^1 f(x)\,dx = \operatorname{Inf} U(P,f) = 1$$

$$\int_0^1 f(x)\,dx = \operatorname{Sup} L(P,f) = -1$$

$$\therefore \quad \int_0^1 f(x)\,dx \neq \int_0^1 f(x)\,dx$$

Hence $f \notin R[0\ 1]$

Example #

If
$$f(x) = \begin{cases} 0 & \text{where } x \text{ is rational} \\ 1 & \text{where } x \text{ is irrational} \end{cases}$$

, then show that f is not integrable in any interval

Sol # Let $P = \{a = x_0, x_1, x_2, \dots x_n = b\}$ be any partition of any interval $[a\ b]$ &

$$I_r = [x_{r-1}, x_r] \quad r = 1, 2 \dots n$$

clearly $M_r = 1$ and $m_r = 0$

$$U(P,f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{i=1}^{n} 1\cdot(x_r - x_{r-1})$$

$$= 1(x_n - x_0) = (b-a)$$

$$L(P,f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} 0.(x_r - x_{r-1})$$

$$= 0.(x_n - x_0) = 0.(b-a) = 0$$

$$\overline{\int_a^b} f(x)\,dx = \operatorname{Inf} U(P,f) = 1$$

$$\underline{\int_a^b} f(x)\,dx = \operatorname{Sup} L(P,f) = 0$$

$$\overline{\int_a^b} f(x)\,dx \neq \underline{\int_a^b} f(x)\,dx$$

Hence $f \notin R[a\ b]$

$\because$ $[a\ b]$ an arbitray interval

$\therefore$ f is not integrable in any interval of real line.

Note # The above two examples are functions bounded but not integrable.

Theorem # The upper and Lower integrals are defined for every bounded function f.

Proof # $\because$ f is bounded function on $[a\ b]$

$\therefore$ It will be bounded above and below on $[a, b]$.

Let M and m be upper and lower bounds of f on $[a, b]$. Then

$$m \leq f(x) \leq M \quad \forall x \in [a\ b]$$

Let $P = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$ be any partition of $[a, b]$ and M_i, m_i respectively be sup and inf of f on $[x_{i-1}, x_i]$. Then

$$M_i \leq M \ \& \ m \leq m_i \quad \forall i$$

Hence

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$$

$$\geq \sum_{i=1}^{n} m \Delta x_i \quad (\because m_i \geq m)$$

$$= m \sum_{i=1}^{n} \Delta x_i$$

$$= m(b-a) \longrightarrow (1)$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$\leq \sum_{i=1}^{n} M \Delta x_i \quad (\because M_i \leq M)$$

$$= M \sum_{i=1}^{n} \Delta x_i$$

$$= M(b-a)$$

$$U(P,f) \leq M(b-a) \longrightarrow ②$$

Also

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i \leq \sum_{i=1}^{n} M_i \Delta x_i$$

$$= U(P,f)$$

$$(\because m_i \leq M_i)$$

$$L(P,f) \leq U(P,f) \longrightarrow ③$$

Combining ① ,② ,③

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

It is clear that this condition holds for all Partitions. Hence the sets $\{U(P,f) : P \in \mathcal{P}[a,b]\}$ and $\{L(P,f) : P \in \mathcal{P}[a,b]\}$ of upper sums and Lower sums are bounded sets and hence attain their sup and inf on $[a\ b]$

$\Rightarrow$ upper and Lower integrals exist for bounded function f

Theorem# If f is a bounded function on $[a, b]$, then

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$$

Proof#

$\because L(P,f) \le U(P,f)$ for all partitions

$\Rightarrow \operatorname{Sup}\{L(P,f) : P \in \mathcal{P}[a,b]\} \le \operatorname{Inf}\{U(P,f) : P \in \mathcal{P}[a,b]\}$

$\Rightarrow \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f\,dx.$

Theorem#

A bounded function $f(x)$ is integrable over $[a.b]$ and M, m are bounds of $f(x)$ over $[a.b]$. Show that

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Proof# We know that

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

for all Partitions

$$\Rightarrow m(b-a) \le \operatorname*{Sup}_P L(P,f) \le \operatorname*{Inf}_P U(P,f) \le M(b-a)$$

$$\Rightarrow m(b-a) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le M(b-a)$$

$\because f(x)$ is integrable

$\therefore \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$

$$\Rightarrow m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

As required.

Remarks # Since we have

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

for all partitions.

There force, we have

$$\text{Sup}\{U(P,f) : P \in \mathcal{P}[a,b]\} = M(b-a)$$

and

$$\text{Inf}\{L(P,f) : P \in \mathcal{P}[a,b]\} = m(b-a)$$

Riemann-Stieltjes Integral w.r.t an Increasing Integrator

Upper Riemann-Stieltjes Sum

Let f be a bounded function on $[a\ b]$, let α be be a function defined and monotonically increasing on $[a, b]$ and let $P = \{a = x_0, x_1, x_2 - - - x_n = b\}$ be a partition of $[a\ b]$. Then upper Stieltjes or Riemann Stieltjes sum of f w.r.t α for the partition P is denoted by $U(P, f, \alpha)$ and is defined by

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i[\alpha(x_i) - \alpha(x_{i-1})]$$

$$= \sum_{i=1}^{n} M_i \Delta\alpha_i$$

where $M_i = lub\{f(x) \mid x \in [x_{i-1}, x_i]\}$

$m_i = Inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$

and $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

Lower Riemann-Stieltjes Sum

Let f be a bounded function on $[a\ b]$, let α be a function defined and monotonically increasing on $[a.b]$ and let $P = \{a = x_0, x_1, x_2 \ldots x_n = b\}$ be any partition of $[a.b]$. The Lower stieltjes or Riemann stieltjes sum of f w.r.t α for the partition P is denoted by $L(P,f,\alpha)$ and is defined by

$$L(P,f\alpha) = \sum_{i=1}^{n} m_i \Delta x_i$$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

$$= \sum_{i=1}^{n} m_i[\alpha(x_i) - \alpha(x_{i-1})]$$

where $m_i = Inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$

and $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$

Note # 1) Actually we are considering these sums w.r.t monotone functions on $[a,b]$ but we have considered monotonically increasing function α. We can do this without losing generality because if α are decreasing, then $\beta = -\alpha$ would be increasing and for every bounded function f and for every partition P. we have

$U(P,f.\alpha) = -U(P.f,\beta)$. Thus any ~~resulting~~ we obtain for increasing integrators will transfer to the class of decreasing functions

2) # we will denote $\alpha(x_i) - \alpha(x_{i-1})$ by $\Delta\alpha_i$. Then number $\Delta\alpha_i$ is the weight assigned

to the Component intervals $[x_{i-1}, x_i]$. The assumption that α is increasing is equivalent to assuming that the weight assigned to each Component interval is non-negative.

3) # Many of the proofs of the results for Riemann integrals will be duplicate of our proofs of the Corresponding facts for Riemann Integrals, with only minor Changes. We will have to replace Δx_i with $\Delta \alpha_i$ and $b-a$ with $\alpha(b) - \alpha(a)$

4) # Most of the elementry properties of the Riemann sums and integrals remain valid for Stieltjes sums and integrals.

We mention one property that fails for Stieltjes integrals. It is possible for f to be integrable with respect to α on intervals $[a, c]$ and $[c, b]$, but not on $[a, b]$. However if either f or α is continuous, this property also holds.

Theorem # The (upper) sets of upper and Lower Riemann Stieltjes sums w.r.t $\alpha \uparrow$ on $[a, b]$ for a bound function f on $[a, b]$ are bounded.

Proof # Let f be a function bounded on $[a, b]$ and α be defined and monotically increasing on $[a, b]$.

Since f is bounded on $[a, b]$, therefore $\exists$ two numbers M, m such that

$$m \leq f(x) \leq M \quad \forall x \in [a, b].$$

Let $P = \{a = x_0, x_1, x_2 \ldots \ldots x_n = b\}$ be a partition of $[a, b]$ and

$m_i = \text{Inf}\{f(x) : x_{i-1} \leq x \leq x_i\}$

$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$.

We have

$$m \le m_i \le M_i \le M$$

$$\Rightarrow \sum_{i=1}^{n} m\Delta\alpha_i \le \sum_{i=1}^{n} m_i\Delta\alpha_i \le \sum_{i=1}^{n} M_i\Delta\alpha_i \le \sum M\Delta\alpha_i$$

$$\Rightarrow m\sum_{i=1}^{n}\Delta\alpha_i \le L(P,f,\alpha) \le U(P,f,\alpha) \le M\sum_{i=1}^{n}\Delta\alpha_i$$

$$\Rightarrow m[\alpha(b)-\alpha(a)] \le L(P,f,\alpha) \le U(P,f,\alpha) \le M[\alpha(b)-\alpha(a)]$$

Clearly this relation holds for all partitions of $[a,b]$

$\Rightarrow$ The sets $\{U(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$ and $\{L(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$ are bounded and attain their sup and infimum. This fact leads us to the definition of upper and lower Riemann Stieltjes integrals.

Properties of $U(P,f,\alpha)$ & $L(P,f,\alpha)$

1) # If f is a function bounded on $[a,b]$ and α is function monotonically increasing on $[a,b]$ and let $P=\{x_0=a, x_1, x_2, \dots x_n=b\}$ be a partition of $[a,b]$. Then $\{U(P,f,\alpha)\}$ & $\{L(P,f,\alpha)\}$ and bounded and

$$U(P,f,\alpha) \ge L(P,f,\alpha) \quad \forall P$$

Proof # Since f is bounded on $[a,b]$, therefore it will be bounded. Let M, m be upper and lower bounds of $f(x)$ over $[a,b]$ i.e

$$m \le f(x) \le M \qquad a \le x \le b$$

If M_i & m_i are sup and inf of $f(x)$ over $[x_{i-1}, x_i]$, then

$$m \le m_i \le M_i \le M$$

$$\Rightarrow \sum_{i=1}^{n} m\Delta\alpha_i \le \sum_{i=1}^{n} m_i\Delta\alpha_i \le \sum_{i=1}^{n} M_i\Delta\alpha_i \le \sum_{i=1}^{n} M\Delta\alpha_i$$

$$\Rightarrow m[\alpha(b)-\alpha(a)] \le L(P,f,\alpha) \le U(P,f,\alpha) \le M(\alpha(b)-\alpha(a))$$

$\because$ This relation holds for all partitions of $[a,b]$

$\therefore$ We have

$$U(P,f,\alpha) \geq L(P,f,\alpha) \quad \forall P$$

Also,

$$m[\alpha(b)-\alpha(a)] \le L(P,f,\alpha) \le M[\alpha(b)-\alpha(a)] \rightarrow (1) \quad \forall P$$

and

$$m[\alpha(b)-\alpha(a)] \le U(P,f,\alpha) \le M[\alpha(b)-\alpha(a)] \quad \forall P \rightarrow (2)$$

From (1) & (2) we conclude that the sets of lower sums and upper sums are bounded.

note that

$$\sum_{i=1}^{n} \Delta\alpha_i = \sum_{i=1}^{n}[\alpha(x_i)-\alpha(x_{i-1})]$$

$$= \alpha_1(x_1) - \alpha_0(x_0) + \cancel{\alpha_2}(x_2) - \cancel{\alpha_1}(x_1) + \alpha_3(x_3) - \cancel{\alpha_2}(x_2) + \cdots + \alpha_n(x_n) - \cancel{\alpha_{n-1}}(x_{n-1})$$

$$= \alpha_n(x_n) - \alpha_0(x_0) = \alpha(b) - \alpha(a)$$

Note # It should be noted that

$$\operatorname{Sup}\{U(P,f,\alpha) : P \in \mathcal{P}[a,b]\} = M[\alpha(b)-\alpha(a)]$$

$$\operatorname{Inf}\{L(P,f,\alpha) : P \in \mathcal{P}[a,b]\} = m[\alpha(b)-\alpha(a)]$$

2) # 17 P_1 and P_2 are any two Partitions of $[a,b]$ and P_2 is a refinement of P_1 i.e

$P_1 \subset P_2$, Then

(a) # $U(P_2, f, \alpha) \leq U(P_1, f, \alpha)$.

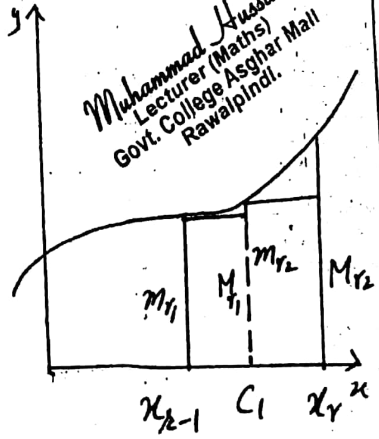
(b) # $L(P_2, f, \alpha) \geq L(P_1, f, \alpha)$

i.e the upper Riemann Stieltjes sum can not increase by the refineness of the partition and the Lower Riemann Stieltjes sum can not decrease by the refinement of partition.

Proof # Let $P_1 = \{a = x_0, x_1, x_2 \cdots x_n = b\}$ and P_2 be a partition with k additional or surplus points $c_1, c_2 \cdots c_k$ than those of P_1.

Let $P_2' = \{c_1\} \cup P_1$ where $x_{r-1} < c_1 < x_r$

Let



$M_r = Sup\{f(x) : x_{r-1} \leq x \leq x_r\}$

$M_{r_1} = Sup\{f(x) : x_{r-1} \leq x \leq c_1\}$

$M_{r_2} = Sup\{f(x) : c_1 \leq x \leq x_r\}$

$m_{r_1} = Inf\{f(x) : x_{r-1} \leq x \leq c_1\}$

$m_{r_2} = Inf\{f(x) : c_1 \leq x \leq x_r\}$

$m_r = Inf\{f(x) : x_{r-1} \leq x \leq x_r\}$

$$Max\{M_{r_1}, M_{r_2}\} \leq M_r$$

$$U(P_1, f\alpha) - U(P_2', f\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i - \Big[\sum_{i=1}^{r-1} M_i \Delta\alpha_i$$

$$+ M_{r_1}\{\alpha(c_1) - \alpha(x_{r-1})\} + M_{r_2}\{\alpha(x_r) - \alpha(c_1)\}$$

$$+ \sum_{i=r+1}^{n} M_i \Delta\alpha_i\Big]$$

$$= \cancel{\sum_{i=1}^{r-1} M_i \Delta\alpha_i} + M_r\{\alpha(x_r) - \alpha(x_{r-1})\} + \cancel{\sum_{i=r+1}^{n} M_i \Delta\alpha_i}$$
$$- \cancel{\sum_{i=1}^{r-1} M_i \Delta\alpha_i} - M_{r_1}\{\alpha(c_1) - \alpha(x_{r-1})\} - M_{r_2}\{\alpha(x_r) - \alpha(c_1)\}$$
$$- \cancel{\sum_{i=r+1}^{n} M_i \Delta\alpha_i}$$

$$= M_r\{\alpha(x_r) - \alpha(x_{r-1})\} - M_{r_1}\{\alpha(c_1) - \alpha(x_{r-1})\}$$
$$- M_{r_2}\{\alpha(x_r) - \alpha(c_1)\}$$

$$= M_r\{\alpha(x_r) + \alpha(c_1) - \alpha(c_1) - \alpha(x_{r-1})\} - M_{r_1}\{\alpha(c_1) - \alpha(x_{r-1})\}$$
$$- M_{r_2}\{\alpha(x_r) - \alpha(c_1)\}$$

$$= M_r\{\alpha(c_1) - \alpha(x_{r-1})\} + M_r\{\alpha(x_r) - \alpha(c_1)\} - M_{r_1}\{\alpha(c_1) - \alpha(x_{r-1})\}$$
$$- M_{r_2}\{\alpha(x_r) - \alpha(c_1)\}$$

$$= (M_r - M_{r_1})\{\alpha(c_1) - \alpha(x_{r-1})\} + (M_r - M_{r_2})\{\alpha(x_r) - \alpha(c_1)\}$$

$\because$ α is monotonically increasing and

$\therefore$ $\alpha(c_1) - \alpha(x_{r-1}) \geq 0$ $\quad$ $\alpha(x_r) - \alpha(c_1) \geq 0$

Also $\because$ $M_r \geq M_{r_1}$ & $M_{r_2} \leq M_r$

$\therefore$ $M_r - M_{r_1} \geq 0$ & $M_r - M_{r_2} \geq 0$

Hence

$U(P_1 f, \alpha) - U(P_2', f, \alpha) \geq 0$ or $U(P_1, f, \alpha) \geq U(P_2', f, \alpha)$

Now taking

$$P_2'' = P_2' \cup \{c_2\} = P_1 \cup \{c_1\} \cup \{c_2\}$$

$$U(P_2', f, \alpha) \geq U(P_2'', f, \alpha)$$

taking $P_2''' = P_2'' \cup \{c_3\} = P_1 \cup \{c_1\} \cup \{c_2\} \cup \{c_3\}$

$$U(P_2'', f, \alpha) \geq U(P_2''', f, \alpha)$$

Repeating this process we reach at

$$P_2^k = P_2^{k-1} \cup \{c_k\} = P_1 \cup \{c_1\} \cup \{c_2\} \cup \dots\dots \cup \{c_k\} = P_2.$$

and.

$$U(P_2^{k-1}, f, \alpha) \geq U(P_2^k, f, \alpha) = U(P_2, f, \alpha)$$

Combining all these inequalities, we have

$$U(P_1, f, \alpha) \geq U(P_2', f, \alpha) \geq U(P_2'', f, \alpha) \geq U(P_2''', f, \alpha) \geq \dots U(P_2^{k-1}, f, \alpha) \geq U(P_2)$$

$$\Rightarrow U(P_1, f, \alpha) \geq U(P_2, f, \alpha) \quad \text{as Required.}$$

OR

The result $U(P_1, f, \alpha) \geq U(P_2', f, \alpha)$ can also be proved as

$$U(P_2', f, \alpha) = \sum_{i=1}^{n+1} M_i \Delta\alpha_i$$

$$= \sum_{i=1}^{r-1} M_i \Delta\alpha_i + M_{r_1}\{\alpha(c_1) - \alpha(x_{r-1})\} + M_{r_2}\{\alpha(x_r) - \alpha(c_1)\} + \sum_{i=r+1}^{n} M_i \Delta\alpha_i$$

$$\leq \sum_{i=1}^{r-1} M_i \Delta\alpha_i + M_r\{\alpha(c_1) - \alpha(x_{r-1})\} + M_r\{\alpha(x_r) - \alpha(c_1)\} + \sum_{i=r+1}^{n} M_i \Delta\alpha_i \quad (\because M_r \geq M_{r_1} \ \& \ M_r \geq M_{r_2})$$

$$= \sum_{i=1}^{r-1} M_i \Delta\alpha_i + M_r\{\alpha(x_r) - \alpha(x_{r-1})\} + \sum_{i=r+1}^{n} M_i \Delta\alpha_i$$

$$= \sum_{i=1}^{n} M_i \Delta\alpha_i = U(P_1, f, \alpha)$$

$$\Rightarrow U(P_1, f, \alpha) \geq U(P_2', f, \alpha)$$

(b) # $L(P_2', f, \alpha) = \sum_{i=1}^{n+1} m_i \Delta\alpha_i$

$$= \sum_{i=1}^{r-1} m_i \Delta\alpha_i + m_{r_1}\{\alpha(c_1) - \alpha(x_{r-1})\} + m_{r_2}\{\alpha(x_r) - \alpha(c_1)\} + \sum_{i=r+1}^{n} m_i \Delta\alpha_i$$

$$\geq \sum_{i=1}^{r-1} m_i \Delta\alpha_i + m_r\{\alpha(c_1) - \alpha(x_{r-1})\} + m_r\{\alpha(x_r) - \alpha(c_1)\} + \sum_{i=r+1}^{n} m_i \Delta\alpha_i$$

$\because \; m_r \leq m_{r_1}$ & $m_r \leq m_{r_2}$

$$= \sum_{i=1}^{r-1} m_i \Delta\alpha_i + m_r\{\alpha(x_r) - \alpha(x_{r-1})\} + \sum_{i=r+1}^{n} m_i \Delta\alpha_i$$

$$= \sum_{i=1}^{n} m_i \Delta\alpha_i = L(P_1, f\alpha)$$

$\Rightarrow L(P_2', f\alpha) \geq L(P_1, f, \alpha)$

Repeating this process for P_2', $P_2'' = P_2' \cup \{c_2\}$

$$L(P_2'', f, \alpha) \geq L(P_2', f, \alpha)$$

Repeating the process for P_2'', $P_3''' = P_2'' \cup \{c_3\}$

$$L(P_3''', f, \alpha) \geq L(P_2'', f, \alpha)$$

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Repeating the process for P_2^{k-1}, $P^k = P^{k-1} \cup \{c_k\} = P_2$

$$L(P_2, f, \alpha) \geq L(P_2^{k-1}, f, \alpha)$$

Combining all these inequalities, we have

$$L(P_1,f,\alpha) \le L(P_2',f,\alpha) \le L(P_2'',f,\alpha) \le \cdots$$
$$\cdots \le L(P_2^{k-1},f,\alpha) \le L(P_2,f,\alpha)$$
$$\Rightarrow L(P_1,f,\alpha) \le L(P_2,f,\alpha)$$

3) #

$$L(P_1,f,\alpha) \le U(P_2,f,\alpha)$$

For any two partitions of $[a,b]$

Proof # Let P be the common refinement of P_1, P_2

i.e

$$P = P_1 \cup P_2$$

Then $\quad P_1 \subset P \;\; \& \;\; P_2 \subset P$

and by the property ② above, we have

$$L(P_1,f,\alpha) \le L(P,f,\alpha) \le U(P,f,\alpha) \le U(P_2,f,\alpha)$$
$$\Rightarrow L(P_1,f,\alpha) \le U(P_2,f,\alpha)$$

Similarly $\quad L(P_2,f,\alpha) \le U(P_1,f,\alpha)$

4 # Estimate of Difference in Sums

If P_1, P_2 are any two partitions of $[a,b]$, $\|P_1\| \le \delta$ (+ve real number); then for

$P_1 \subset P_2$ with p additional points

and $|f(x)| < k \;\; \forall x \in [a,b]$, we have

$$U(P_1,f,\alpha) \le U(P_2,f,\alpha) + 2pk\delta$$

Proof # Let $P_1 = \{a = x_0, x_1, x_2, \ldots x_n = b\}$ and P_2 has additional points $c_1, c_2, \ldots c_p$ than those of P_1.

Let $P_2' = \{c_1\} \cup P_1$ where $x_{r-1} < c_1 < x_r$

x_{r-1} — c_1 — x_r

Let

$M_{r_1} = \sup\{f(x) : x_{r-1} \le x \le c_1\}$

$M_{r_2} = \sup\{f(x) : c_1 \le x \le x_r\}$

Then

$$U(P_1) - U(P_2') = \sum_{i=1}^{n} M_i \Delta\alpha_i - \Big[\sum_{i=1}^{r-1} M_i \Delta\alpha_i + M_{r_1}\{\alpha(c_1) - \alpha(x_{r-1})\} + M_{r_2}\{\alpha(x_r) - \alpha(c_1)\} + \sum_{i=r+1}^{n} M_i \Delta\alpha_i\Big]$$

$$= (M_r - M_{r_1})\{\alpha(c_1) - \alpha(x_{r-1})\} + (M_r - M_{r_2})\{\alpha(x_r) - \alpha(c_1)\}$$

$\because \; |f(x)| < k \quad \forall x \in [a,b] \longrightarrow (1)$

$\Rightarrow -k < f(x) < k \quad \forall x \in [a,b]$

$\Rightarrow -k \le \begin{matrix} M_{r_1} \\ M_{r_2} \end{matrix} \le M_r \le k$

$\Rightarrow 0 \le \begin{matrix} M_r - M_{r_1} \\ M_r - M_{r_2} \end{matrix} \le 2k$

using in (1)

$$U(P_1, f, \alpha) - U(P_2', f, \alpha) \le 2k\,[\alpha(c_1) - \alpha(x_{r-1})] + 2k\,[\alpha(x_r) - \alpha(c_1)]$$

$$= 2k\,[\alpha(x_r) - \alpha(x_{r-1})] = 2k\,\Delta\alpha_r$$

$U(P_1,f,\alpha) - U(P_2',f,\alpha) = 2k\Delta\alpha_r$

$\leq 2k\delta$ $\quad \because \|P\| \leq \delta$

$\Delta\alpha_r \leq \delta$

$U(P_1,f,\alpha) \leq U(P_2',f,\alpha) + 2k\delta$

Repeating the above process with P_2' & $P_2'' = P_2' \cup \{c_2\}$ we have

$U(P_2',f,\alpha) \leq U(P_2'',f,\alpha) + 2k\delta$

Similarly for P_2'', P_3'''

$U(P_2'',f,\alpha) \leq U(P_3''',f,\alpha) + 2k\delta$

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For $\quad P_2^{p-1}$, $P^p = P_1 \cup \{c_1, c_2 \dots c_p\} = P_2$

$U(P_2^{p-1},f,\alpha) \leq U(P_2,f,\alpha) + 2k\delta$

Adding all these p equations and inequalities

$U(P_1,f,\alpha) + U(P_2',f,\alpha) + U(P_2'',f,\alpha) + \dots U(P_2^{p-1},f,\alpha)$

$\leq U(P_2',f,\alpha) + U(P_2'',f,\alpha) + \dots U(P_2^{p-1},f,\alpha) + U(P_2,f,\alpha) + 2pk\delta$

$\Rightarrow U(P_1,f,\alpha) \leq U(P_2,f,\alpha) + 2pk\delta$

Similarly it can be proved that

$L(P_2,f,\alpha) \leq L(P_1,f,\alpha) + 2kp\delta$

By Muhammad Hussain Lecturer (Maths)
Govt-College Asghar Mall RWP

5) # **Theorem** # Let f_1, f_2 be bounded function on $[a\ b]$ and P be a partition of $[a\ b]$. Then

(a) # $U(P, f_1+f_2) \leq U(P,f_1)+U(P,f_2)$

(b) # $L(P, f_1+f_2) \geq L(P, f_1, \alpha)+L(P, f_2, \alpha)$

Proof # Let $f = f_1+f_2$

$\because$ f_1, f_2 are bounded on $[a\ b]$

$\therefore$ f is also bounded on $[a\ b]$

Let $M_i = \text{Sup}\, f(x) \quad x_{i-1} \leq x \leq x_i$

$m_i = \text{Inf}\, f(x) \quad x_{i-1} \leq x \leq x_i$

$\omega_{i_1} = \text{Inf}\, f_1 \quad x_{i-1} \leq x \leq x_i \qquad \omega_{i_2} = \text{Inf}\, f_2(x) \quad x_{i-1} \leq x \leq x_i$

$W_{i_1} = \text{Sup}\, f_1 \quad x_{i-1} \leq x \leq x_i \qquad W_{i_2} = \text{Sup}\, f_2(x) \quad x_{i-1} \leq x \leq x_i$

Then $f_1(x) \leq W_{i_1} \quad \forall x \in [x_{i-1}, x_i]$

$f_2(x) \leq W_{i_2} \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow f_1(x)+f_2(x) \leq W_{i_1}+W_{i_2} \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow f(x) \leq W_{i_1}+W_{i_2} \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow M_i \leq W_{i_1}+W_{i_2} \quad \forall$ " " " $\longrightarrow (1)$

$f_1(x) \geq \omega_{i_1} \quad \forall x \in [x_{i-1}\ x_i]$

$f_2(x) \geq \omega_{i_2} \quad \forall x \in [x_{i-1}\ x_i]$

$\Rightarrow f_1(x)+f_2(x) \geq \omega_{i_1}+\omega_{i_2} \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow f(x) \geq \omega_{i_1}+\omega_{i_2}$ " " "

$\Rightarrow m_i \geq \omega_{i_1}+\omega_{i_2}$ " " $\longrightarrow (2)$

From ①

$$\sum_{i=1}^{n} M_i \Delta\alpha_i \leq \sum_{i=1}^{n} W_{i_1} \Delta\alpha_i + \sum_{i=1}^{n} W_{i_2} \Delta\alpha_i$$

$$\Rightarrow U(P,f,\alpha) \leq U(P,f_1,\alpha) + U(P,f_2,\alpha)$$

As Required.

From ②

$$\sum_{i=1}^{n} m_i \Delta\alpha_i \geq \sum_{i=1}^{n} \omega_{i_1} \Delta\alpha_i + \sum_{i=1}^{n} \omega_{i_2} \Delta\alpha_i$$

$$\Rightarrow L(P,f,\alpha) \geq L(P,f_1,\alpha) + L(P,f_2,\alpha)$$

As Required

Riemann Stieltjes Integral

(Definition)

Let f be a real valued function defined and bounded over $[a,b]$. Let $P = \{a = x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[a,b]$.

Let α be a monotonically increasing function function on $[a,b]$ (Since $\alpha(a)$, $\alpha(b)$ are finite, it follows that α is bounded on $[a,b]$).

Corresponding to each partition P of $[a,b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

The upper sum of f, w.r.t α for partition P is

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

where $M_i = lub\{f(x) \mid x \in [x_{i-1}, x_i]\}$

The Lower sum of f w.r.t α for the partition P is

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

where $m_i = glb\{f(x) \mid x \in [x_{i-1}, x_i]\}$

of course, these sums exist because f is bounded. Also the sets of upper and lower sums are bounded. The upper Riemann Stieltjes integral w.r.t α on $[a, b]$ is defined.

$$\overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f(x)\,d\alpha(x) = Inf\{\cancel{f(x)}\,U(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$$

The lower Riemann Stieltjes Integral w.r.t α on $[a\ b]$ is defined as

$$\underline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f(x)\,d\alpha = Sup\{L(P,f,\alpha) : P \in \mathcal{P}[a.b]\}$$

If the upper and lower integrals are equal, then the common value of these integrals is called Riemann Stieltjes integral or simply Stieltjes integal and is written as

$$\int_a^b f\,d\alpha \text{ or } \int_a^b f(x)\,d\alpha(x)$$

If $\int_a^b f\,d\alpha$ exists, we say that f is Riemann integrable function w.r.t α on $[a.b]$ and is symbolically written as

$f \in R(\alpha, [a,b])$ or $R_\alpha[a,b]$ or $R(\alpha)$ over $[a\ b]$ or RS_α on $[a\ b]$ or $f \in RS[\alpha : a, b]$

Remarks # Riemann stieltjes integral is a general case of Riemann integral because if $\alpha(x) = x$, then $\Delta\alpha_i = \Delta x_i$ and the

and the Stieltjes integral reduces to Riemann integral. In other words Riemann integral is a special case of the Stieltjes integral.

Since the integral $\int_a^b f(x)\,d\alpha(x)$ depends upon f, α, a, b but not on the variable x of integration, we may omit the variable and write

$\int_a^b f\,d\alpha$ instead of $\int_a^b f(x)\,d\alpha(x)$. The letter x is a "dummy variable"

Example (a)# If $\alpha(x) = k \quad \forall x \in [a,b]$ and f is any bounded function on $[a\ b]$, then prove that

$$f \in R(\alpha)$$

(b) # If α is an increasing function on $[a\ .b]$ and $f(x) = k$, then prove that
$f \in R(\alpha)$.

Sol # Let P be any partition of $[a.\ b]$.
Then $\alpha(x) = k \quad \forall x \in [x_{i-1}, x_i]$

and
$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$= \sum_{i=1}^{n} m_i[\alpha(x_i) - \alpha(x_{i-1})]$$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

$$= \sum_{i=1}^{n} m_i(k-k) = 0$$

$\Rightarrow$ Every lower sum is zero

Hence

$$\underline{\int_a^b} f\, d\alpha = \sup_P L(P,f,\alpha) = 0$$

Again

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$= \sum_{i=1}^{n} M_i [\alpha(x_i) - \alpha(x_{i-1})]$$

$$= \sum_{i=1}^{n} M_i (k-k) = 0$$

$\Rightarrow U(P,f,\alpha) = 0 \quad \forall P$

$\Rightarrow$ Every upper sum is zero

Hence $\overline{\int_a^b} f\, d\alpha = \inf_P L(P,f,\alpha) = 0$

Thus $\underline{\int_a^b} f\, d\alpha = \overline{\int_a^b} f\, d\alpha$

$\Rightarrow f \in R(\alpha)$

(b) $\because f(x) = k \quad \forall x \in [a,b]$

$\Rightarrow m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$
$= k \quad \forall P$

and $M_i = k \quad \forall P$ and $\forall i$

Hence $U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$

$= k\sum_{i=1}^{n} \Delta\alpha_i = k[\alpha(x_n) - \alpha(x_0)]$

$= k[\alpha(b) - \alpha(a)]$

$$U(P,f,\alpha) = k[\alpha(b)-\alpha(a)] \quad \forall P$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \inf_P U(P,f,\alpha) = k[\alpha(b)-\alpha(a)]$$

Similarly

$$\underline{\int_a^b} f\,d\alpha = k[\alpha(b)-\alpha(a)]$$

Thus

$$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$$

$$\Rightarrow f \in R(\alpha)$$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mali
Rawalpindi.

Theorem # The upper and Lower Riemann Stieltjes integrals are always defined for a bounded function f

Proof # Let f be a bounded function on $[a, b]$, let α be an increasing function on $[a, b]$. Suppose M, m are upper and lower bounds of $f(x)$ in $[a, b]$ i.e

$$m \leq f(x) \leq M \quad \forall x \in [a, b]$$

Let M_i, m_i denote the sup and inf of $f(x)$ in the interval $[x_{i-1}, x_i]$ for a certain partition P of $[a, b]$. Then

$$M_i \leq M \quad \forall i = 1, 2, \ldots n$$

$$m \leq m_i \quad \forall i = 1, 2, \ldots n$$

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i$$

$$\geq \sum_{i=1}^{n} m \Delta \alpha_i \qquad \because m_i \geq m \ \forall i$$

$$\Rightarrow L(P,f,\alpha) \geq m \sum_{i=1}^{n} \Delta\alpha_i$$

$$= m[\alpha(x_1) - \alpha(x_0) + \alpha(x_2) - \alpha(x_1) + \cdots \alpha(x_n) - \alpha(x_{n-1})]$$

$$= m[\alpha(x_n) - \alpha(x_0)]$$

$$= m[\alpha(b) - \alpha(a)]$$

$$L(P,f,\alpha) \geq m[\alpha(b) - \alpha(a)] \longrightarrow ①$$

Similarly $U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$

$$\leq \sum_{i=1}^{n} M \Delta\alpha_i \quad \because M_i \leq M \ \forall i$$

$$= M[\alpha(b) - \alpha(a)]$$

$$\Rightarrow U(P,f,\alpha) \leq M[\alpha(b) - \alpha(a)] \longrightarrow ②$$

Also

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \leq \sum_{i=1}^{n} M_i \Delta\alpha_i$$

$$L(P,f,\alpha) \leq U(P,f,\alpha) \quad \because M_i \geq m_i \longrightarrow ③$$

Combining ①, ② & ③

$$m[\alpha(b) - \alpha(a)] \leq L(P,f,\alpha) \leq U(P,f,\alpha) \leq M[\alpha(b) - \alpha(a)]$$

This condition is true for all partitions

$\Rightarrow$ The upper sums and lower sums form bounded sets and attain their sup and inf.

$\Rightarrow$ upper and lower integrals are always defined for a bounded function.

Theorem# Let f be a fn defined and bounded on $[a\ b]$ and $\alpha(x)$ be a monotonically increasing function. Then

$$\sup_P L(P,f,\alpha) \leq \inf_P U(P,f,\alpha)$$

or

$$\sup\{L(P,f,\alpha) : P \in \mathcal{P}[a,b]\} \leq \inf\{U(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$$

$$\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

Proof# Let P^* be the common refinement of P_1 & P_2 on $[a,b]$. Then

$$L(P_1,f,\alpha) \leq L(P^*,f,\alpha) \longrightarrow ①$$

$$U(P_2,f,\alpha) \geq U(P^*,f,\alpha) \longrightarrow ②$$

$$L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \longrightarrow ③$$

Combining these results

$$L(P_1,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P_2,f,\alpha)$$

$$\Rightarrow L(P_1,f,\alpha) \leq U(P_2,f,\alpha)$$

If P_1 is fixed and Inf is taken over all partitions P_2, then

$$L(P_1,f,\alpha) \leq \inf U(P_2,f,\alpha)$$

$$\Rightarrow \quad L(P_1,f,\alpha) \leq \overline{\int_a^b} f\,d\alpha$$

Similarly if Sup is taken over all partitions P_1

We get $\int_{\underline{a}}^{b} f d\alpha \leq \int_{a}^{\overline{b}} f d\alpha$

OR

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

$\Rightarrow$ $U(P_2, f, \alpha)$ is an upper bound for the set of lower sums because this condition is true for any two partitions P_1, P_2 and if P_2 is fixed and partitions are changed in place of P_1, then lower sums will remain less or equal to $U(P_2, f, \alpha)$. Thus

$$\int_{\underline{a}}^{b} f d\alpha \leq U(P_2, f, \alpha)$$

$\Rightarrow$ $\int_{\underline{a}}^{b} f d\alpha$ is a lower bound for the set of upper sums and hence

$$\int_{\underline{a}}^{b} f d\alpha \leq \int_{a}^{\overline{b}} f d\alpha$$

Theorem# If f is a bounded function on $[a, b]$ and $\alpha(x)$ is a $\uparrow$ function on $[a, b]$, then for every $\epsilon > 0$, $\exists$ $\delta > 0$ (depending upon ϵ) such that

(a) # $U(P, f, \alpha) < \int_{a}^{\overline{b}} f d\alpha + \epsilon$ $\forall$ partitions P with $\|P\| \leq \delta$

(b) # $L(P, f, \alpha) > \int_{\underline{a}}^{b} f d\alpha - \epsilon$ $\forall$ partitions with $\|P\| \leq \delta$

Proof# (a) Since

$$\int_{a}^{\overline{b}} f d\alpha = \operatorname{Inf}\{U(P, f, \alpha) : P \in \mathcal{P}[a, b]\}$$

Therefore for $\epsilon > 0$, $\exists$ a partition P_1 such that

$$U(P_1, f, \alpha) < \int_{a}^{\overline{b}} f d\alpha + \epsilon \longrightarrow (1)$$

$\because$

$$\int_{\underline{a}}^{b} f d\alpha = \operatorname{Sup}\{L(P, f, \alpha) : P \in \mathcal{P}[a, b]\}$$

$\therefore$ For $\epsilon > 0$, $\exists$ a partition P_2 such that

$$L(P_2, f, \alpha) > \int_{-a}^{b} f d\alpha - \epsilon \longrightarrow (2)$$

Let P^* be a refinement of P_1, P_2. Then

$$U(P^*, f, \alpha) \leq U(P_1, f, \alpha)$$

$$L(P^*, f, \alpha) \geq L(P_2, f, \alpha)$$

Then

$$U(P^*, f, \alpha) < \int_{a}^{-b} f d\alpha + \epsilon \longrightarrow (3)$$

$$L(P^*, f, \alpha) > \int_{-a}^{b} f d\alpha - \epsilon \longrightarrow (4)$$

Let $\|P^*\| = \delta > 0$

Then we can write

$$U(P, f, \alpha) < \int_{a}^{-b} f d\alpha + \epsilon \quad \forall P, \text{ where } \|P\| \leq \|P^*\| = \delta$$

$$L(P, f, \alpha) > \int_{-a}^{b} f d\alpha - \epsilon \quad \forall P, \|P\| \leq \delta.$$

Theorem # (Cauchy's Criterion for integrability OR Riemann Condition)

Let f be a bounded function on $[a, b]$ and α be an increasing function on $[a, b]$. Then $f \in R(\alpha)$ iff for every $\epsilon > 0$, however small, there exists a partition P of $[a\ b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

i.e. difference between upper and Lower sums remain arbitrarily small.

Proof# Let $f \in R(\alpha)$ and let $\epsilon > 0$ be given.

Then $$\int_a^b f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha.$$

$\because$ $$\overline{\int_a^b} f\,d\alpha = \text{Inf}\{U(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$$

$\therefore$ By definition of infimum there exists a partition P_1, such that
$$\overline{\int_a^b} f\,d\alpha + \epsilon/2 > U(P_1,f,\alpha) \longrightarrow ①$$

Similarly since
$$\underline{\int_a^b} f\,d\alpha = \text{Sup}\{L(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$$

Therefore $\exists$ a partition P_2 such that
$$\underline{\int_a^b} f\,d\alpha - \epsilon/2 < L(P_2,f,\alpha) \longrightarrow ②$$

Adding ① & ② we get
$$U(P_1,f,\alpha) + \underline{\int_a^b} f\,d\alpha - \epsilon/2 < L(P_2,f,\alpha) + \overline{\int_a^b} f\,d\alpha + \epsilon/2$$

$\because$ $$\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$$

$$\therefore U(P_1,f,\alpha) + \int_a^b f\,d\alpha < L(P_2,f,\alpha) + \int_a^b f\,d\alpha + \epsilon$$

$$\Rightarrow U(P_1,f,\alpha) - L(P_2,f,\alpha) < \epsilon.$$

OR

Let $P = P_1 \cup P_2$

$$U(P,f,\alpha) - L(P,f,\alpha) \le U(P_1,f,\alpha) - L(P_2,f,\alpha)$$

$\because \; U(P,f,\alpha) \leq U(P_1,f,\alpha) \rightarrow (1)$

$L(P,f,\alpha) \geq L(P_2,f,\alpha)$

$\Rightarrow -L(P,f,\alpha) \leq -L(P_2,f,\alpha) \rightarrow (2)$

Adding (1) & (2)

$$U(P,f,\alpha) - L(P,f,\alpha) \leq U(P_1,f,\alpha) - L(P_2,f,\alpha)$$

$$U(P,f,\alpha) - L(P,f,\alpha) \leq U(P_1,f,\alpha) - L(P_2,f,\alpha)$$

$$< \overline{\int_a^b} f\,d\alpha + \epsilon/2 - \left(\underline{\int_a^b} f\,d\alpha + \epsilon/2\right)$$

$$= \epsilon \qquad \because \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

So Condition is proved.

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

Converse

Conversely suppose that for every $\epsilon > 0$, $\exists$ a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

We are to prove that $f \in R(\alpha)$

$$\because \; \underline{\int_a^b} f\,d\alpha \geq L(P,f,\alpha) \quad \forall P$$

$$\Rightarrow \underline{\int_a^b} f\,d\alpha - L(P,f,\alpha) \geq 0 \longrightarrow (3)$$

Also $\because \; U(P,f,\alpha) \geq \overline{\int_a^b} f\,d\alpha \quad \forall P$

$$\Rightarrow U(P,f,\alpha) - \overline{\int_a^b} f\,d\alpha \geq 0 \longrightarrow (4)$$

Adding (3) & (4) we have

$$\int_{\underline{a}}^{b} f\,d\alpha - L(P,f,\alpha) + U(P,f,\alpha) - \int_{a}^{\bar{b}} f\,d\alpha \geq 0$$

$$\Rightarrow -\left[\int_{a}^{\bar{b}} f\,d\alpha - \int_{\underline{a}}^{b} f\,d\alpha\right] \geq -U(P,f,\alpha) + L(P,f,\alpha)$$

$$\Rightarrow \int_{a}^{\bar{b}} f\,d\alpha - \int_{\underline{a}}^{b} f\,d\alpha \leq U(P,f,\alpha) - L(P,f,\alpha)$$

$$\because \int_{a}^{\bar{b}} f\,d\alpha \geq \int_{\underline{a}}^{b} f\,d\alpha$$

$$\Rightarrow 0 \leq \int_{a}^{\bar{b}} f\,d\alpha - \int_{\underline{a}}^{b} f\,d\alpha < \epsilon \qquad \forall\, \epsilon > 0$$

This can be satisfied for every $\epsilon > 0$ only if

$$\int_{a}^{\bar{b}} f\,d\alpha - \int_{\underline{a}}^{b} f\,d\alpha = 0$$

$$\int_{a}^{\bar{b}} f\,d\alpha = \int_{\underline{a}}^{b} f\,d\alpha$$

$$\Rightarrow f \in R(\alpha)$$

Theorem # If $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ holds for some P and some ϵ, then

(a)# It holds (with same ϵ) for every refinement of P

(b)# If $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ holds for partition $P = \{a = x_0, x_1, x_2 - - - - - x_n = b\}$ and if s_i, t_i arbitrary points in $[x_{i-1}, x_i]$

Then

$$\sum_{i=1}^{n} | f(s_i) - f(t_i) | \Delta\alpha_i < \epsilon.$$

<u>Proof</u># (a) Let P^* be a refinement of P, then

$$L(P, f, \alpha) \leq L(P^*, f, \alpha) \longrightarrow ①$$

$$U(P^*, f, \alpha) \leq U(P, f, \alpha) \longrightarrow ②$$

Adding ① & ②

$$L(P, f, \alpha) + U(P^*, f, \alpha) \leq L(P^*, f, \alpha) + U(P, f, \alpha)$$

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) \leq \underbrace{U(P, f, \alpha) - L(P, f, \alpha)}_{< \epsilon}$$

$$\Rightarrow U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon$$

$\Rightarrow$ The result should hold for any refinement of P.

(b) # Let $\sup f(x) = M_i \quad \forall x \in [x_{i-1}, x_i]$

$\inf f(x) = m_i \quad \forall x \in [x_{i-1}, x_i]$

Since s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, therefore both $f(s_i)$ & $f(t_i)$ lie in $[m_i, M_i]$ i.e

$$m_i \leq f(s_i) \leq M_i \longrightarrow ①$$

and

$$m_i \leq f(t_i) \leq M_i \longrightarrow ②$$

$$\Rightarrow \quad 0 \leq f(s_i) \leq M_i - m_i \longrightarrow ③$$

$$0 \leq f(t_i) \leq M_i - m_i \longrightarrow ④$$

$$\Rightarrow 0 \leq f(s_i) - f(t_i) \leq M_i - m_i$$

$$\Rightarrow |f(s_i) - f(t_i)| \leq M_i - m_i \quad \forall i$$

$$\Rightarrow \sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i \leq (M_i - m_i)\Delta\alpha_i$$

$$= \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$\Rightarrow \sum_{i=1}^{n} |f(s_i) - f(t_i)| \leq U(P, f, \alpha) - L(P, f, \alpha)$$

$$< \epsilon$$

$$\Rightarrow \sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$$

Theorem # If f is bounded on $[a, b]$ and α be increasing on $[a\ b]$, then $f \in R(\alpha)$ iff for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ (depending upon ϵ) such that

$$\left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\, d\alpha \right| < \epsilon$$

For every partition $P = \{x_0, x_1, x_2, \dots x_n\}$ with $\|P\| < \delta$ and for every choice of $t_i \in [x_{i-1}\ x_i]$

OR $\because$ a continuous function on $[a\ b]$ is bounded

$\therefore$ In statement above we may take a continuous function on $[a\ b]$

Proof # Let $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$ be any partition of $[a, b]$ and

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$$

$$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$$

Then for every choice of t_i in $[x_{i-1}, x_i]$, we have

$$L(P, f, \alpha) \leq \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \leq U(P, f, \alpha) \longrightarrow (1)$$

Let $f \in R(\alpha)$ and $\epsilon > 0$. Then Riemann's Condition holds and there exists a partition P^* of $[a, b]$ s.t. that

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon$$

Let P be a partition of $[a, b]$ such that

$$P^* \subset P \quad \text{and} \quad \|P\| \leq \delta.$$

For this refinement of P^*, we have

$$U(P, f, \alpha) \geq U(P^*, f, \alpha) \rightarrow (a)$$

and

$$L(P^*, f, \alpha) \geq L(P, f, \alpha)$$

$$\Rightarrow L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

$$\Rightarrow -L(P, f, \alpha) \geq -L(P^*, f, \alpha) \rightarrow (b)$$

Adding (a) & (b)

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$$U(P, f, \alpha) \leq U(P^*, f, \alpha) \longrightarrow (a)$$

and

$$L(P, f, \alpha) \geq L(P^*, f, \alpha)$$

$$-L(P,f,\alpha) \leq -L(P^*,f,\alpha) \longrightarrow (b)$$

Adding (a) & (b)

$$U(P,f,\alpha) - L(P,f,\alpha) \leq U(P^*,f,\alpha) - L(P^*,f,\alpha)$$

$$\leq \epsilon \quad \text{by } ①$$

$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon \longrightarrow ②$$

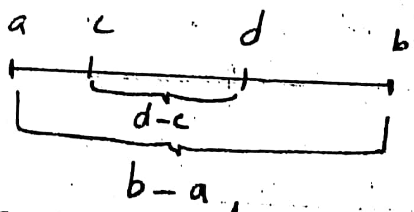
Also for the same partition P

$$L(P,f,\alpha) \leq \int_a^b f\,d\alpha \leq U(P,f,\alpha) \longrightarrow ③$$

and for the same partition P; we have

$$L(P,f,\alpha) \leq \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i \leq U(P,f,\alpha) \longrightarrow ④$$

Now when two real numbers belong to the same interval, their +ve difference lies in the length of interval i.e in the difference of the end points of the interval. Therefore by ③ & ④, we have



$$\left|\sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_a^b f\,d\alpha\right| \leq U(P,f,\alpha) - L(P,f,\alpha)$$

$$< \epsilon \quad \text{by } ②$$

Hence the condition is proved.

OR

The condition can also be proved as under

Let $f \in R(\alpha)$. Then we have

$$\overline{\int_a^b} f\,d\alpha = \int_{\underline{a}}^{b} f\,d\alpha = \int_a^b f\,d\alpha$$

Let $\epsilon > 0$. By Darboux's Theorem $\exists\ \delta > 0$ such that for every partition P with $\|P\| \leq \delta$

$$U(P,f,\alpha) < \overline{\int_a^b} f\,d\alpha + \epsilon = \int_a^b f\,d\alpha + \epsilon \longrightarrow ①$$

and

$$L(P,f,\alpha) > \int_{\underline{a}}^{b} f\,d\alpha - \epsilon = \int_a^b f\,d\alpha - \epsilon \longrightarrow ②$$

If t_i are any points the intervals $[x_{i-1}, x_i]$ of P, then we have

$$L(P,f,\alpha) \leq \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i \leq U(P,f,\alpha) \longrightarrow ③$$

From ①, ② and ③ we deduce that for every partition P with $\|P\| \leq \delta$

$$\int_a^b f\,d\alpha - \epsilon < L(P,f,\alpha) \leq \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i \leq U(P,f,\alpha) < \int_a^b f\,d\alpha + \epsilon$$

$$\Rightarrow \int_a^b f\,d\alpha - \epsilon < \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i < \int_a^b f\,d\alpha + \epsilon$$

$$\Rightarrow -\epsilon < \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_a^b f\,d\alpha < \epsilon$$

$$\Rightarrow \left| \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_a^b f\,d\alpha \right| < \epsilon$$

As required.

Converse # Conversely suppose that for any real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha\right| < \epsilon$$

for every partition $P = \{x_0, x_1, x_2 \dots x_n\}$ with $\|P\| \leq \delta$ and for every choice of points t_i in $[x_{i-1}, x_i]$

We prove that $f \in R(\alpha)$

$\because$ The condition holds

$\therefore$ For all choices t_i, t_i' in $[x_{i-1}, x_i]$ and for all partitions P with $\|P\| \leq \delta$, we have

$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha\right| \leq \epsilon/4 \longrightarrow ①$$

$$\left|\sum_{i=1}^{n} f(t_i')\Delta\alpha_i - \int_a^b f\,d\alpha\right| \leq \epsilon/4 \longrightarrow ②$$

Since

$$M_i - m_i = \sup\{f(x) - f(x') : x, x' \in [x_{i-1}, x_i]\}$$

Therefore by definition of Supremum, we have

$$M_i - m_i - \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} < f(t_i) - f(t_i')$$

$$\Rightarrow M_i - m_i < f(t_i) - f(t_i') + \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \longrightarrow ③$$

$U(P,f,\alpha)-L(P,f,\alpha)$

$$= \sum_{i=1}^{n}(M_i-m_i)\Delta\alpha_i$$

$$< \sum_{i=1}^{n}\left(f(t_i)-f(t_i')\right)\Delta\alpha_i + \frac{\epsilon}{2[\alpha(b)-\alpha(a)]}\sum\Delta\alpha_i$$

$$= \sum_{i=1}^{n}\left[f(t_i)-f(t_i')\right]\Delta\alpha_i + \frac{\epsilon}{2[\alpha(b)-\alpha(a)]}[\alpha(b)-\alpha(a)]$$

$$= \sum_{i=1}^{n}\left[f(t_i)-f(t_i')\right]\Delta\alpha_i + \frac{\epsilon}{2}$$

$$= \sum_{i=1}^{n}f(t_i)\Delta\alpha_i - \sum_{i=1}^{n}f(t_i')\Delta\alpha_i + \epsilon/2$$

$$= \sum_{i=1}^{n}f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha - \sum_{i=1}^{n}f(t_i')\Delta\alpha_i + \int_a^b f\,d\alpha + \epsilon/2$$

$$\leq \left|\sum_{i=1}^{n}f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha - \left(\sum f(t_i')\Delta\alpha_i - \int_a^b f\,d\alpha\right)\right| + \epsilon/2$$

$$\leq \left|\sum_{i=1}^{n}f(t_i)\Delta\alpha_i - \int_a^b f\,d\alpha\right| + \left|\sum_{i=1}^{n}f(t_i')\Delta\alpha_i - \int_a^b f\,d\alpha\right| + \epsilon/2$$

$$< \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon/2 + \epsilon/2 = \epsilon$$

$\Rightarrow f \in R(\alpha)$

Theorem # Let f, α be bounded on $[a, b]$ with α increasing. The following are equivalent.

(a) # $f \in R(\alpha)$

(b) # Riemann's Condition holds for f w.r.t α on $[a, b]$

(c) # $[L\alpha(P, f, \alpha) = U(P, f, \alpha)]$: $\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$

Proof# (a) $\Longrightarrow$ (b)

Let $f \in R_\alpha[a, b]$. Then

$$\int_a^b f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

Fix $\epsilon > 0$. Then by Darboux's theorem $\exists\ \delta > 0$ such that for every partition P with $\|P\| \leq \delta$, we have

$$U(P, f, \alpha) < \overline{\int_a^b} f\,d\alpha + \epsilon/3 = \int_a^b f\,d\alpha + \epsilon/3 \longrightarrow (1)$$

and

$$L(P, f, \alpha) > \underline{\int_a^b} f\,d\alpha - \epsilon/3 = \int_a^b f\,d\alpha - \epsilon/3 \longrightarrow (2)$$

If t_i are points in the partition intervals $[x_{i-1}, x_i]$, then we have

$$L(P, f, \alpha) \leq \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \leq U(P, f, \alpha) \longrightarrow (3)$$

From (1), (2) and (3) we deduce that for every partition P with $\|P\| \leq \delta$

$$\int_a^b f\,d\alpha - \epsilon/3 < L(P, f, \alpha) \leq \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \leq U(P, f, \alpha) < \int_a^b f\,d\alpha + \epsilon/3$$

$$\Rightarrow \int_a^b f\,d\alpha - \epsilon/3 \leq \sum_{i=1}^{n} f(t_i)\Delta\alpha_i < \int_a^b f\,d\alpha + \epsilon/3$$

$$\Rightarrow -\epsilon/3 \le \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha < \epsilon/3$$

$$\Rightarrow \left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \right| < \epsilon/3 \longrightarrow (4)$$

Similarly we can choose points t_i' in $[x_{i-1}, x_i]$ such that

$$\left| \sum_{i=1}^{n} f(t_i')\Delta\alpha_i - \int_a^b f d\alpha \right| < \epsilon/3 \longrightarrow (5)$$

For any such choices of t_i, t_i', we have

$$\left| \sum_{i=1}^{n} [f(t_i) - f(t_i')]\Delta\alpha_i \right|$$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

$$= \left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \sum_{i=1}^{n} f(t_i')\Delta\alpha_i \right|$$

$$= \left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i + \int_a^b f d\alpha - \int_a^b f d\alpha - \sum_{i=1}^{n} f(t_i')\Delta\alpha_i \right|$$

$$\le \left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \right| + \left| \sum_{i=1}^{n} f(t_i')\Delta\alpha_i - \int_a^b f d\alpha \right|$$

$$< \epsilon/3 + \epsilon/3 = 2\epsilon/3 \longrightarrow (5)$$

Let $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$ $i = 1, 2, 3 \ldots n$

and $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$ $i = 1, 2, 3 \ldots n$

Then by definition of sup and inf, we can choos t_i and t_i' in $[x_{i-1}, x_i]$ such that

$$M_i - \frac{\epsilon}{3[\alpha(b)-\alpha(a)]} < f(t_i) \leq M_i$$

and

$$m_i \leq f(t_i') < m_i + \frac{\epsilon}{3[\alpha(b)-\alpha(a)]}$$

Therefore

$$M_i - m_i - \frac{\epsilon}{3[\alpha(b)-\alpha(a)]} < f(t_i) - f(t_i')$$

Multiplying by $\Delta\alpha_i > 0$ and summing over i

$$\sum_{i=1}^{n}[M_i\Delta\alpha_i - m_i\Delta\alpha_i] - \frac{\epsilon}{3[\alpha(b)-\alpha(a)]}\sum_{i=1}^{n}\Delta\alpha_i$$

$$< \sum_{i=1}^{n}[f(t_i) - f(t_i')]\Delta\alpha_i$$

$$\Rightarrow \sum_{i=1}^{n} M_i\Delta\alpha - \sum_{i=1}^{n} m_i\Delta\alpha_i - \frac{\epsilon}{3} < \sum_{i=1}^{n}[f(t_i) - f(t_i')]\Delta\alpha_i$$

$$U(P,f,\alpha) - L(P,f,\alpha) - \epsilon/3 < \sum_{i=1}^{n}[f(t_i) - f(t_i')]\Delta\alpha_i$$

$$< \frac{2\epsilon}{3} \quad \text{by (5)}$$

$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) - \epsilon/3 < 2\epsilon/3$$

$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \frac{2\epsilon}{3} + \epsilon/3 = \epsilon$$

$\Rightarrow$ Riemann's Condition is true

Thus (a) $\Longrightarrow$ (b)

(b) $\Longrightarrow$ (c)

Suppose that Riemann's Condition holds for f on [a.b]. We must prove that

$$U(P,f,\alpha) = L(P,f,\alpha)$$

Fix $\epsilon > 0$. Since Riemann's Condition holds for every $\epsilon > 0$, therefore there exists a partition P_o such that

$$U(P_o f,\alpha) - L(P_o f,\alpha) < \epsilon$$

Then for any refinement P of P_o, we have

$$U(P,f,\alpha) \leq U(P_o f,\alpha) \longrightarrow (a)$$

and

$$L(P,f,\alpha) \geq L(P_o,f,\alpha) \longrightarrow$$

$$\Rightarrow \quad -L(P,f,\alpha) \leq -L(P_o,f,\alpha) \longrightarrow (b)$$

Adding (a) & (b)

$$U(P,f,\alpha) - L(P,f,\alpha) \leq U(P_o,f,\alpha) - L(P_o,f,\alpha) < \epsilon$$

$$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon \longrightarrow ⑥$$

$$\Rightarrow 0 \leq U(P,f,\alpha) - L(P,f,\alpha) < \epsilon \quad \text{for every } \epsilon > 0$$

$$\Rightarrow \quad U(P,f,\alpha) = L(P,f,\alpha)$$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mal
Rawalpindi.

$$\Rightarrow (b) \Longrightarrow (c)$$

Next we prove that (c) $\Longrightarrow$ (a)

We assume that $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$ and prove that $f \in R(\alpha)$.

Let $\quad \underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = I$

$\because \quad I = \underline{\int_a^b} f d\alpha = \sup\{L(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$

$\therefore$ we can choose a partition P_1 such that

$$I - \epsilon < L(P_1, f, \alpha) \rightarrow (7)$$

$\because \quad I = \overline{\int_a^b} f d\alpha = \operatorname{Inf}\{U(P,f,\alpha) : P \in \mathcal{P}[a,b]\}$

$\therefore$ we can choose a partition P_2 such that

$$U(P_2, f, \alpha) < I + \epsilon \rightarrow (8)$$

let $P_0 = P_1 \cup P_2$. Then.

$$I - \epsilon < L(P_1, f, \alpha) \leq L(P_0, f, \alpha) \rightarrow (9)$$

and

$$U(P_0, f, \alpha) \leq U(P_2, f, \alpha) < I + \epsilon \rightarrow (10)$$

$\Rightarrow$ By (9) & (10) for any refinement P of P_0 i.e $\|P\| \leq \|P_0\|$ and for any choices t_i in $[x_{i-1}, x_i]$ for P, we have

$$I - \epsilon < L(P_1, f, \alpha) \leq L(P, f, \alpha) \leq \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \leq U(P, f, \alpha)$$
$$\leq U(P_2, f, \alpha) < I + \epsilon$$

$$\Rightarrow I - \epsilon < \sum_{i=1}^{n} f(t_i)\Delta\alpha_i < I + \epsilon$$

$$-\epsilon < \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - I < \epsilon$$

$$\Rightarrow \left| \sum_{i=1}^{n} f(t_i)\Delta\alpha_i - I \right| < \epsilon$$

for every P, with $\|P\| \leq \|P_0\| = \delta$

Thus for any $\epsilon > 0$, we have found a number $\delta = \|P_0\|$ such that

$$\left| \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - I \right| < \epsilon$$

For every partition P with $\|P\| \leq \delta$ and for every choice of $t_i \in [x_{i-1}, x_i]$

This proves that $\int_a^b f\,d\alpha$ exists and is equal to I

$\Rightarrow \quad f \in R(\alpha)$

Riemann-Stieltjes Sums

Let f and α be bounded on $[a, b]$. Let $P = \{a = x_0, x_1, x_2, \ldots x_n\}$ be a partition of $[a, b]$. Let $T = \{t_1, t_2, \ldots t_n\}$ where $t_i \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots n$. A sum of the form

$$S(P, f, \alpha) = \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i$$

is called Riemann's Stieltjes sum w.r.t α for the partition P. It is also denoted by

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

$$S(P, T, f)$$

Let f, α be bounded on $[a, b]$ with α increasing. Then clearly

$$L(P, f) \leq S(f, P, T) \leq U(P, f)$$

For any points T. Thus if the upper and lower sums approximate the integral, we would see

that the Riemann–Stieltjes sums would also approximate the integral. The next theorem shows that this is the case

Note # In general with no restriction on the function f and α, the numerical values of $S(f,\alpha,P)$ may vary widely. However, under suitable restrictions on f, and α, the values of $S(f,\alpha,P)$ will cluster near some unique number I whenever P is sufficiently refined

Theorem # Let f be a bounded function on $[a,b]$. Then $f \in R_\alpha[a,b]$ iff there exists a number I having the property that for every $\epsilon > 0$, there a partition P such that for every refinement P^* of P and for every choice of points t_i in the partition intervals $[x_{i-1}, x_i]$ of P^*, we have

$$|S(f,\alpha,P^*) - I| < \epsilon$$

Proof # Suppose $f \in R_\alpha[a,b]$. Then Riemann's condition holds and for $\epsilon > 0$ there exists a partition P such that

$$U(f,\alpha,P) - L(f,\alpha,P) < \epsilon$$

Let P^* be a refinement of P. Then

$$U(f,\alpha,P^*) \leq U(f,\alpha,P) \quad \rightarrow (1)$$

and

$$L(f,\alpha,P^*) \geq L(f,\alpha,P) \quad \rightarrow (2)$$

$$\Rightarrow \quad -L(f,\alpha,P^*) \leq -L(f,\alpha,P) \quad \rightarrow (3)$$

Adding ① & ③

$$U(P^*,f,\alpha) - L(P^*,f,\alpha) \leq U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

$$\Rightarrow U(P^*,f,\alpha) - L(P^*,f,\alpha) < \epsilon \longrightarrow ④$$

Also

$$L(P^*,f,\alpha) \leq S(P^*,f,\alpha) \leq U(P^*,f,\alpha)$$

and

$$L(P^*,f,\alpha) \leq \int_a^b f\,d\alpha \leq U(P^*,f,\alpha) \longrightarrow ⑤$$

$$\Rightarrow 0 \leq \left|S(P^*,f,\alpha) - \int_a^b f\,d\alpha\right| \leq U(P^*,f,\alpha) - L(P^*,f,\alpha) < \epsilon$$

$$\Rightarrow \left|S(P^*,f,\alpha) - \int_a^b f\,d\alpha\right| < \epsilon$$

It follows that $I = \int_a^b f\,d\alpha$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

Converse # Now suppose the condition holds. We may assume that $\alpha(b) > \alpha(a)$. Let $\epsilon > 0$. There exists a partition $P = \{x_0, x_1, x_2, \dots, x_n\}$ of $[a,b]$ such that

$$|S(f,P,T) - I| < \epsilon/4$$

for all points $T = \{t_1, t_2, \dots, t_n\}$ in intervals $[x_{i-1}, x_i]$ of P

$$\text{Let } M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$$

Then by definition of supremum, for each i, $1 \leq i \leq n$, there exists $t_i^* \in [x_{i-1}, x_i]$ such that

$$M_i - \frac{\epsilon}{4[\alpha(b) - \alpha(a)]} < f(t_i^*)$$

$$\Rightarrow \sum_{i=1}^{n}\left(M_i - \frac{\epsilon}{4[\alpha(b)-\alpha(a)]}\right)\Delta\alpha_i \leq \sum_{i=1}^{n} f(t_i^*)\Delta\alpha_i$$

$$\Rightarrow \sum_{i=1}^{n} M_i\Delta\alpha_i - \frac{\epsilon}{4[\alpha(b)-\alpha(a)]}\sum_{i=1}^{n}\Delta\alpha_i \leq S(f,P,T^*)$$

where $T^* = \{t_1^*, t_2^*, \ldots t_n^*\}$

$$\Rightarrow U(P,f,\alpha) - \frac{\epsilon}{4[\alpha(b)-\alpha(a)]}[\alpha(b)-\alpha(a)] \leq S(f,P,T^*)$$

$$\Rightarrow U(P,f,\alpha) - \frac{\epsilon}{4} \leq S(f,P,T^*) \longrightarrow ⑥$$

Similarly, there exists set of points T^{**} such that

$$S(f,P,T^{**}) \leq L(P,f,\alpha) + \epsilon/4 \longrightarrow ⑦$$

Therefore $\Rightarrow -L(P,f,\alpha) \leq -S(f,P,T^{**}) + \epsilon/4$

$$U(P,f,\alpha) - L(P,f,\alpha) \leq \left[S(f,P,T^*) + \epsilon/4\right] + \left[-S(f,P,T^{**}) + \epsilon/4\right]$$

$$\leq S(f,P,T^*) - S(f,P,T^{**}) + \epsilon/2.$$

$$\leq S(f,P,T^*) - I + I - S(f,P,T^{**}) + \epsilon/2$$

$$\leq |S(f,P,T^*) - I| + |S(f,P,T^{**}) - I| + \epsilon/2$$

$$< \epsilon/4 + \epsilon/4 + \epsilon/2 = \epsilon$$

$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

$\Rightarrow$ Riemann's Condition is true

Hence $f \in R_\alpha[a,b]$

Remarks (1) # The condition $P^* \supset P$ in the above theorem is analogous to the condition " $n \geq N$ " in the definition of the limit of a sequence.

(2) # The above theorem shows that refining partitions is a way of forcing convergence of Riemann–Stieltjes sums. Another way to describe convergence of Riemann–Stieltjes sums is given in the next definition.

Definition # Let $P = \{x_0, x_1, x_2 \dots x_n\}$ be a partition of $[a, b]$. We define

$$\text{norm } P = \max\{|x_i - x_{i-1}| : 1 \leq i \leq n\}$$

Let f and α be bounded functions on $[a, b]$. We say that

$$\lim_{\text{norm } P \to 0} S(f, P, T) = I$$

If for every $\epsilon > 0$, there exists $\delta > 0$ such that for any partition P of $[a, b]$ with $\|P\| < \delta$ and for any points T, we have

$$|S(f, P, T) - I| < \epsilon$$

Remarks # We can not prove a theorem analogous to above theorem for the kind of convergence described in in Definition above i.e convergence of Riemann Stieltjes sums as described in Definition above is not equivalent to the existence of the Riemann–Stieltjes integral.

For example. Let

$$f(x) = \begin{cases} 0 & -1 \leq x < 0 \\ 1 & 0 \leq x \leq 1 \end{cases}$$

and $\alpha(x) = \begin{cases} 0 & -1 \leq x \leq 0 \\ 1 & 0 < x \leq 1 \end{cases}$

Let $P = \{-1, 0, 1\}$

Then $U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$

$= M_1 \Delta\alpha_1 + M_2 \Delta\alpha_2$

$= 0\,[\alpha(0) - \alpha(-1)] + 1\cdot[\alpha(1) - \alpha(0)]$

$= 0 + 1[1 - 0]$

$= 1$

$L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$

$= m_1[\alpha(0) - \alpha(-1)] + m_2[\alpha(1) - \alpha(0)]$

$= -1[0 - 0] + 1[1 - 0]$

$= 1$

and thus $f \in R_\alpha[-1, 1]$

However $\lim_{\|P\| \to 0} S(f, P, T)$ does not exist

If $P^* \supset P = \{-1, 0, 1\}$, then

$1 \leq L(f, P) \leq L(f, P^*) \leq S(f, P^*, T) \leq U(f, P^*)$

and therefore $\leq U(f, P) = 1$

$S(f, P^*, T) = 1$

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Thus there partitions P^* of $[a, b]$ with arbitrary small norm such that

$$S(f; P^*, T) = 1$$

On the other hand if

$$P^* = \left\{ -\frac{n}{n}, -\frac{n-1}{n}, \ldots, -\frac{2}{n}, -\frac{1}{n}, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} \right\}$$

and we choose $t = -\frac{1}{n}$ in the intervals $[-\frac{1}{n}, \frac{1}{n}]$ we have

$$S(f; P^*, T) = \frac{1}{n}\left[f(-1) + f\left(-\frac{n-1}{n}\right) + \ldots f\left(-\frac{1}{n}\right) + \cancel{f(\ldots) + f\left(\frac{n-1}{n}\right)} \right]$$

$$= \frac{1}{n}[0 + 0 + \ldots + 0 + 0 + 0]$$

$$= 0$$

and norm of $P^* = \frac{2}{n}$.

Thus there are partitions P^* of $[a, b]$ with arbitrarily small norm such that

$$S(f, P^*, T) = 0$$

It follows that $\lim\limits_{\|P\| \to 0} S(f, P, T)$ does not exist.

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

In spite of the previous example, we do have the following Two theorems.

Theorem # Let f be a bounded function on $[a, b]$ and let α be an increasing function on $[a, b]$. If $\lim\limits_{\|P\| \to 0} S(f, P, T)$ exists, then $f \in R_\alpha[a, b]$ and

$$\lim_{\|P\| \to 0} S(f, P, T) = \int_a^b f \, d\alpha$$

Proof # Let $\epsilon > 0$, there exists $\delta > 0$ such that $\|P\| < \delta$, then

$$|S(f, P, T) - I| < \epsilon$$

Let P^* be a partition of $[a, b]$ such that norm $P < \delta$. If $P^* \supset P$, then $\|P^*\| < \delta$ and thus

$$|S(f, P^*, T) - I| < \epsilon$$

$$\Rightarrow f \in R_\alpha[a, b] \text{ and } I = \int_a^b f\,d\alpha.$$

by above theorem.

~~**Theorem #** Let f be a bounded function on $[a, b]$ and α be an increasing function on $[a, b]$~~ (Cancelled)

Let α be an increasing function on $[a, b]$ and suppose that $f \in R_\alpha[a, b]$. If either f or α is continuous on $[a, b]$, then

$$\int_a^b f\,d\alpha = \lim_{\|P\| \to 0} S(f, P, T)$$

Proof # First suppose that f is continuous on $[a, b]$. We may assume that $\alpha(b) > \alpha(a)$. Let $\epsilon > 0$. Since f is continuous on $[a, b]$, and $[a, b]$ being closed and bounded is a compact set, continuous image of a compact set is compact, therefore image set under f will be also compact. Now the compact set is bounded, therefore f is bounded. Again a continuous function on a compact set is

uniformly continuous on $[a, b]$. Thus there exists a $\delta > 0$ corresponding to an arbitrary $\epsilon > 0$ such that

$$|f(s) - f(t)| < \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \quad \text{if } |s-t| \leq \delta.$$

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition with norm $P < \delta$

Let $P = \{x_0, x_1, x_2, \ldots x_n\}$ be a partitions with norm $P < \delta$. Let

$$S(f, P, T) = \sum_{i=1}^{n} f(t_i) \Delta\alpha_i \rightarrow (1)$$

$$\int_a^b f\,d\alpha = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f\,d\alpha = \sum_{i=1}^{n} f(t_i^*) \Delta\alpha_i$$

where $t_i^* \in [x_{i-1}, x_i]$ for $i = 1, 2, \ldots n$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mal
Rawalpindi.

$$\left| S(f, P, T) - \int_a^b f\,d\alpha \right|$$

$$= \left| \sum [f(t_i) - f(t_i^*)] \Delta\alpha_i \right|$$

$$\leq \sum_{i=1} \left| f(t_i) - f(t_i^*) \right| \Delta\alpha_i$$

$$\leq \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \sum_{i=1}^{n} \Delta\alpha_i = \frac{\epsilon}{2} < \epsilon$$

Hence $\lim_{\|P\| \to 0} S(f, P, T) = \int_a^b f\,d\alpha$

Now suppose α is continuous on $[a, b]$. We first assume that $f(x) > 0$ for x in $[a, b]$

Let $\epsilon > 0$. There exists a partition $Q = \{y_0, y_1, \cdots y_n\}$ of $[a, b]$ such that

$$U(f, Q) < \int_a^b f d\alpha + \frac{\epsilon}{4}$$

Let $M = lub\{f(x) : x \in [a, b]\}$.

Now α being continuous on compact set $[a, b]$ is uniformly continuous on $[a\ b]$ and thus there exists $\delta' > 0$ such that if $P = \{x_0, x_1, x_2 \cdots x_j\}$ is a partition of $[a\ b]$ with norm $P < \delta'$, then we have

$$|\alpha(x_i) - \alpha(x_{i-1})| < \frac{\epsilon}{4Mn} \qquad |x_i - x_{i-1}| < \delta'$$

$$\Rightarrow \Delta\alpha_i < \frac{\epsilon}{4Mn} \qquad (\because \Delta\alpha_i > 0)$$

$$\forall i = 1, 2, \cdots j$$

Let A denote the set of integers i for which $1 \leq i \leq j$ and (x_{i-1}, x_i) contains a point of Q. Let

$$B = \{1, 2, \cdots j\} \setminus A$$

Now

$$U(f, P, \alpha) = \sum_{i \in A} M_i \Delta\alpha_i + \sum_{i \in B} M_i \Delta\alpha_i$$

$$\leq (n-1) M \frac{\epsilon}{4Mn} + U(f, Q)$$

$$< \int_a^b f d\alpha + \frac{\epsilon}{2}$$

Now let f be any function in $R_\alpha[a, b]$ and let $\epsilon > 0$. Choose a number k such that $f(x) + k > 0$ on $[a, b]$. By the above argument, there exists $\delta_1 > 0$

Such that if norm $P < \delta$, then

$$U(f+k, P) < \int_a^b (f+k)\,d\alpha + \epsilon/2 \rightarrow ①$$

Since $U(f+k, P) = U(f, P) + k(b-a) \rightarrow ②$

and

$$\int_a^b (f+k)\,d\alpha = \int_a^b f\,d\alpha + k(b-a) \rightarrow ③$$

We find that

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mal
Rawalpindi.

$$U(f, P) + k(b-a) < \int_a^b f\,d\alpha + k(b-a) + \epsilon/2$$

$$\Rightarrow \quad U(f, P) < \int_a^b f\,d\alpha + \epsilon/2 \quad \text{if } \|P\| < \delta_1 \longrightarrow ④$$

Applying the above argument to $-f$, we find that there exists $\delta_2 > 0$ such that if norm $P < \delta_2$, then

$$U(-f, P) < \int_a^b (-f)\,d\alpha + \epsilon/2.$$

Since $U(-f, P) = -L(f, P)$, we conclude that

$$-L(f, P) < -\int_a^b f\,d\alpha + \epsilon/2.$$

$$\Rightarrow L(f, P) > \int_a^b f\,d\alpha - \epsilon/2 \longrightarrow ⑤ \quad \text{if } \|P\| < \delta_2$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if norm $P < \delta$, then

$$\int_a^b f\,d\alpha - \epsilon/2 < L(f, P) \leq S(f, P, T) \leq U(f, P, T) < \int_a^b f\,d\alpha + \epsilon/2$$

$$\Rightarrow -\epsilon/2 < S(f,P,T) - \int_a^b f\,d\alpha < \epsilon/2$$

$$\Rightarrow \left| S(f,P,T) - \int_a^b f\,d\alpha \right| < \epsilon$$

Therefore $\lim\limits_{\|P\|\to 0} S(f,P,T) = \int_a^b f\,d\alpha.$

Corollary # If $f \in R[a,b]$, then

$$\lim_{\|P\|\to 0} S(f,P,T) = \int_a^b f\,dx$$

Proof # Since $\alpha(x) = x$ is continuous, the conclusion follows immediately from the above Theorem.

Example # Let $[a,b]$ be a closed interval and let $c \in (a,b)$. Let $k_1 < k_2$ and define

$$\alpha(x) = \begin{cases} k_1 & a \le x < c \\ k_2 & c < x \le b \end{cases}$$

Let f be bounded function on $[a\ b]$ which is continuous at c. Show that $f \in R(\alpha)$

Solution # Let $\epsilon > 0$

Since f is continuous at c, therefore there exists points x_1 & x_2 with

$a < x_1 < c < x_2 < b$ such that if $x \in [x_1, x_2]$ we have

$$|f(x)-f(c)| < \frac{\epsilon}{3(k_2-k_1)}$$

$$\Rightarrow f(c)-\frac{\epsilon}{3(k_2-k_1)} < f(x) < f(c)+\frac{\epsilon}{3(k_2-k_1)} \to ①$$

Let $P=\{a, x_1, x_2, b\}$ and m_2, M_2 be infimum and supremum of f on the interval $[x_1, x_2]$

Then it follows that for $a < x_1 < c < x_2 < b$

$$f(c)-\frac{\epsilon}{3(k_2-k_1)} \leq m_2 \leq M_2 \leq f(c)+\frac{\epsilon}{3(k_2-k_1)}$$

$$\Rightarrow M_2(k_2-k_1) \leq f(c) \leq f(c)\cdot(k_2-k_1)+\epsilon/3 \to ②$$

$$U(P,f,\alpha)=\sum_{i=1}^{2} M_i\Delta\alpha_i = M_1[\alpha(x_1)-\alpha(a)]+M_2[\alpha(x_2)-\alpha(x_1)] + M_3[\alpha(b)-\alpha(x_2)]$$

$$= M_1[k_1-k_1]+M_2[\alpha(x_2)-\alpha(x_1)]+M_3[k_2-k_2]$$

$$= M_2[\alpha(x_2)-\alpha(x_1)]$$

$$= M_2[k_2-k_1] \leq f(c) \leq f(c)(k_2-k_1)+\frac{\epsilon}{3}$$

$$\Rightarrow U(P,f,\alpha) \leq f(c)(k_2-k_1)+\epsilon/3 \to ③$$

Similarly

$$f(c)\cdot(k_2-k_1)-\epsilon/3 \leq L(P,f,\alpha)$$

$$\Rightarrow -L(P,f,\alpha) \leq -f(c)(k_2-k_1)+\epsilon/3 \to ④$$

Adding ③ & ④

$$U(P,f,\alpha)-L(P,f,\alpha) \leq \epsilon/3+\epsilon/3 = \frac{2}{3}\epsilon < \epsilon$$

≥ 0

$\Rightarrow$ Riemann's Condition is true

$\Rightarrow f \in R_\alpha[a\ b]$.

Now $\int_a^b f d\alpha \leq U(P,f,\alpha) \leq f(c)(k_2-k_1)+\epsilon/3$ $\forall \epsilon$

hence $\int_a^b f d\alpha \leq f(c)(k_2-k_1) \rightarrow (5)$

Similarly $\int_a^b f d\alpha \geq f(c)(k_2-k_1) \longrightarrow (6)$

By (5) & (6), we have

$$\int_a^b f d\alpha = f(c)(k_2-k_1)$$

As required.

Theorem# If f is continuous on $[a\ b]$ and α is increasing on $[a\ b]$, Then $f \in R_\alpha[a\ b]$

Proof# $\because$ any closed and bounded set is Compact

$\therefore [a\ b]$ is a compact set

Again any Continuous mapping of a Compact metric space into a metric space is uniformly Continuous.

Since f is continuous on $[a, b]$ and $[a\ b]$ is a Compact set, therefore f is uniformly continuous on $[a\ b]$. So by definition of uniform Continuity for $\epsilon > 0$ $\exists$ a $\delta > 0$ such that

$$|f(x')-f(x'')| < \frac{\epsilon}{2[\alpha(b)-\alpha(a)]} \longrightarrow (1)$$

whenever $|x'-x''|<\delta$

Since f is continuous on a compact set $[a,b]$, therefore image set under f will also be compact and therefore bounded. In particular f will be bounded on each of sub-interval obtained by the partition P of of $[a,b]$.

Choose a partition $P = \{a = x_0, x_1, x_2 \dots x_n\}$ such that length of each sub-interval $[x_{i-1}, x_i]$ is less than δ i.e $\|P\| < \delta$

$\because$ f is bounded on each sub-interval $[x_{i-1}, x_i]$

$\therefore$ $\exists$ nos x_i' x_i'' belonging to $[x_{i-1}, x_i]$ such that

$$f(x_i') = m_i$$

$$f(x_i'') = M_i$$

$\because$ The length of each sub-interval is less than δ

$\therefore$ $|x_i' - x_i''| < \delta \quad \forall i$

By virtue of ①, we have

$$|f(x_i'') - f(x_i)| < \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \quad \forall i$$

$$\Rightarrow |M_i - m_i| < \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \quad \forall i$$

$$\Rightarrow M_i - m_i < \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \quad \because M_i \geq m_i$$

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$\leq \frac{\epsilon}{2[\alpha(b) - \alpha(a)]} \sum_{i=1}^{n} \Delta\alpha_i$$

$$= \frac{\epsilon}{2} < \epsilon$$

$\Rightarrow U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$

$\Rightarrow$ Riemann's Condition is true.

$\Rightarrow f \in R_\alpha[a, b]$

Method-II

Let $\epsilon > 0$ be given.

Choose $\eta > 0$ so that

$$[\alpha(b) - \alpha(a)]\eta < \epsilon$$

Since f is uniformly continuous on $[a, b]$, therefore for $\eta > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \eta \quad \forall x, y \in [a, b] \text{ with } |x-y| < \delta \quad \longrightarrow (1)$$

Choose a partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ such that length of each component interval of P is less than δ i.e $\|P\| < \delta$.

Then

$$|x_i - x_{i-1}| < \delta \quad \forall i$$

$\Rightarrow |f(x_i) - f(x_{i-1})| < \eta \quad \forall i$ by (1)

$\Rightarrow |M_i - m_i| < \eta \quad \forall i$

$\Rightarrow M_i - m_i < \eta \quad \forall i \quad \because M_i \geq m_i$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$< \eta \sum_{i=1}^{n} \Delta\alpha_i$$

$$= \eta[\alpha(b) - \alpha(a)]$$
$$< \epsilon$$

$\Rightarrow U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$

$\Rightarrow$ For any arbitrary $\epsilon > 0$; we have found a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

$\Rightarrow$ Riemann's Condition holds and hence $f \in R_\alpha[a,b]$

Concepts # 1) The following statements about a set S are equivalent.

(a) # S is Compact (b) # S is closed and bounded

(c) # Every infinite subset of S has a limit point in S.

(2) # A function Continuous on a Compact metric into any metric space is uniformly Continuous

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Theorem # Let f be a monotonic function on $[a,b]$ and $\alpha(x)$ be Continuous on $[a,b]$, Then $f \in R(\alpha)$ (It is, of course still assumed that α is monotonic)

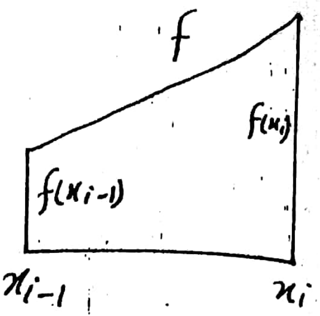
Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mal
Rawalpindi.

OR

If f, α are monotonic and α is Continuous on $[a,b]$, then $f \in R(\alpha)$

Proof # Let f be monotonically increasing and $P = \{a = x_0, x_1, x_2, x_3, \ldots, x_n = b\}$ be any partition of $[a,b]$.

We suppose that f is monotically increasing. Then



$M_i = f(x_i)$ $\quad m_i = f(x_{i-1})$

$= \sup f(x)$ on $[x_{i-1}, x_i]$

$m_i = f(x_{i-1}) = \inf f(x)$ on $[x_{i-1}, x_i]$

(The proof is similar in case f is monotonically decreasing)

$\because$ $\alpha(x)$ is continuous and monotonically increasing

$\therefore$ $\alpha(a) \neq \alpha(b)$ i.e $\alpha(a) < \alpha(b)$

$\Rightarrow$ α assumes every value between a & b or on $[a, b]$

$\Rightarrow$ We can choose $\Delta\alpha_i$ in such a way that each $\Delta\alpha_i$ is same i.e

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{n}$$ where n is +ve integer.

Let $\epsilon > 0$

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$= \frac{\alpha(b) - \alpha(a)}{n}\left[\sum_{i=1}^{n} (f(x_i) - f(x_{i-1})\right]$$

$$= \frac{\alpha(b) - \alpha(a)}{n}[f(b) - f(a)] \longrightarrow ①$$

Now $\dfrac{\alpha(b) - \alpha(a)}{n}[f(b) - f(a)] < \epsilon$

if $\dfrac{\alpha(b) - \alpha(a)}{\epsilon}[f(b) - f(a)] < n$

if $n > \dfrac{\alpha(b) - \alpha(a)}{\epsilon}[f(b) - f(a)] = n_1$

if no of points in partition $P > n_1$

$\Rightarrow$ we can always choose such a number n traced by

$$n_1 = \frac{\alpha(b)-\alpha(a)}{\epsilon}[f(b)-f(a)]$$

from a given $\epsilon > 0$ such that

$$\frac{\alpha(b)-\alpha(a)}{n}[f(b)-f(a)] < \epsilon$$

So from ①

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$ when n is taken large enough.

Thus for every $\epsilon > 0$ we can choose a partition P containing large enough no of points of division such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

$\Rightarrow$ Riemann Condition is true

$\Rightarrow$ $f \in R(\alpha)$

Similarly the case for f be decreasing follows.

Description # We have a theorem that if a a function f is continuous real function on $[a,b]$ and if $f(a) < f(b)$ and if λ is any number such that $f(a) < \lambda < f(b)$, then $\exists$ a point $x \in]a,b[$ such that

$f(x) = c$ i.e f assumes every value between $f(a)$ & $f(b)$

Since α is continuous and monotonically increasing on $[a,b]$, therefore α assumes every value between $\alpha(a)$ & $\alpha(b)$ and we can choose $\Delta\alpha_i$ such that

$$\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$$

Theorem # Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a.b]$ and α is continuous at every point at which f is discontinuous. Then $f \in R(\alpha)$

Proof # Let $\epsilon > 0$. Put $M = \sup|f(x)|$. Let $E = \{a_1, a_2, a_3 \cdots a_p\}$ be the set of points at which f is discontinuous. Since E is finite and α is continuous at every point of E, therefore we can cover E by finitely many disjoint intervals $[u_j, v_j] \subset [a, b]$ such that the sum of corresponding differences is less than ϵ. Further we can enclose the points of E in the interior of intervals $[u_j, v_j]$.

Removing the segments (u_j, v_j) from $[a, b]$ we get a set K of remaining points which is compact. Hence f is uniformly continuous on K and there exists a $\delta > 0$ such that

$$|f(s) - f(t)| < \epsilon \quad \forall s, t \in K, \ |s - t| < \delta \quad \longrightarrow (1)$$

We form a partition $P = \{x_0, x_1, x_2 \cdots\cdots x_n\}$ of $[a, b]$ in such a way that P contains each u_j, and v_j but contains no point any segment (u_j, v_j), and if x_{i-1} is not one of the u_j, then $\Delta x_i < \delta$.

Now

Since
$$|f(x)| \leq M \quad \forall x \in [a, b]$$
$$\Rightarrow -M \leq f(x) \leq M \quad \forall x \in [a, b]$$
$$\Rightarrow 0 \leq f(x) \leq 2M \quad \forall x \in [a, b]$$
$$\Rightarrow M_i \leq 2M \quad \forall i$$
and $m_i \leq 2M \quad \forall i$

and hence
$$M_i - m_i \leq 2M \quad \forall i \quad \longrightarrow (2)$$

Also if x_{i-1} is not one of u_j, then

$$\Delta x_i = x_i - x_{i-1} = |x_i - x_{i-1}| < \delta$$

and by virtue of ① for such intervals, we have

$$M_i - m_i < \epsilon$$

and for intervals for which x_{i-1} is one of u_j, the next end point will be some v_j and for the points of such intervals ① is not true eventhough distance between points is less than δ because points of such intervals are not in k on which f is uniform continuous. Hence for such intervals we have

$$M_i - m_i \leq 2M$$

Let A be the set of indices i for intervals for which 1st end point point is not one of u_j and B be the set all those indices for which the intervals $[x_{i-1}, x_i]$ has some u_j as 1st end point. Then

$$U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i \in A} (M_i - m_i)\Delta\alpha_i + \sum_{i \in B} (M_i - m_i)\Delta\alpha_i$$

$$< \epsilon \sum_{i \in B} \Delta\alpha_i + \sum_{i \in B} 2M\, \Delta\alpha_i$$

$$= \epsilon\,[\alpha(b) - \alpha(a)] + 2M \sum_{i \in B} \Delta\alpha_i$$

$(\because$ B contains indices of intervals of the form $[u_j, v_j]$ on which $\sum \alpha(v_j) - \alpha(u_j) < \epsilon$

$\therefore \sum_{i \in B} \Delta\alpha_i < \epsilon$

$$< \epsilon\,[\alpha(b) - \alpha(a)] + 2M\epsilon = \epsilon'$$

$\because$ ϵ is an arbitrary

$\therefore$ ϵ' depending upon ϵ will also be arbitrary

$\Rightarrow$ For any arbitrary $\epsilon' > 0$, we have a partition P such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon'$$

So Riemann's Condition holds and $f \in R_\alpha$

Riemann-Stieltjes Integration w.r.t An arbitrary Integrator

We extend the definition of the Riemann Stieltjes integral $\int_a^b f d\alpha$ to an arbitray (not necessarily increasing) function α.

Let f, α be bounded functions on $[a\ b]$. We say that f is Riemann stieltjes - integrable w.r.t α on $[a\ b]$ if there exists a number I having property that for every $\epsilon > 0$ there exists a partition P_ϵ of $[a\ b]$ such that for every refinement P of P_ϵ and for every choice of points t_i in the partition intervals $[x_{i-1}, x_i]$, we have

$$\left| \sum_{i=1}^{n} f(t_i) \Delta\alpha_i - I \right| < \epsilon$$

or

$$| S(f, \alpha, P) - I | < \epsilon$$

or

$$| S(f, P, T) - I | < \epsilon$$

for any points $T = \{t_1, t_2 \ldots t_n\}$ in $[x_{i-1}, x_i]$

We denote this number I as $\int_a^b f d\alpha$

Properties of Riemann-Stieltjes Integrals#

1)# Theorem# (Linearity Properties)

(a)# Let f & α be bounded on $[a,b]$ and c be any Constant. If $f \in R(\alpha)$, then $cf \in R(\alpha)$ and

$$\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha.$$

Proof# $\because f \in R(\alpha)$ Case I: For $c > 0$

$\therefore$ for every $\epsilon > 0$ $\exists$ a partition P of $[a,b]$ such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon/c$$

Let $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$

$m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$

Then

$cM_i = \sup\{cf(x) : x_{i-1} \le x \le x_i\}$

$cm_i = \inf\{cf(x) : x_{i-1} \le x \le x_i\}$

$$U(P,cf,\alpha) = \sum_{i=1}^{n} cM_i \Delta\alpha_i = c\sum_{i=1}^{n} M_i\Delta\alpha_i$$

$$= c\,U(P,f,\alpha)$$

and

$$L(P,cf,\alpha) = \sum_{i=1}^{n} cm_i\Delta\alpha_i = c\sum_{i=1}^{n} m_i\Delta\alpha_i$$

$$= c\,L(P,f,\alpha)$$

$$U(P,cf,\alpha) - L(P,cf,\alpha)$$

$$= c[U(P,f,\alpha) - L(P,f,\alpha)]$$

$$< c\cdot\epsilon/c = \epsilon$$

$\because$ ϵ is an arbitrary +ve real number
$\therefore$ For every $\epsilon > 0$, $\exists$ a partition P of $[a, b]$ such that

$$U(P, cf, \alpha) - L(P, cf, \alpha) < \epsilon$$

$\Rightarrow$ Riemann's Condition is true
$\Rightarrow$ $cf \in R(\alpha)$

Also Since $U(P, cf, \alpha) = c\, U(P, f, \alpha)$

Therefore $\inf\limits_{P} U(P, cf, \alpha) = c \inf\limits_{P} U(P, f, \alpha)$

$$\overline{\int_a^b} (cf)\, d\alpha = c \overline{\int_a^b} f\, d\alpha \longrightarrow ①$$

Again since

$$\sup_{P} L(P, cf, \alpha) = c \sup_{P} L(P, f, \alpha)$$

$$\Rightarrow \underline{\int_a^b} cf\, d\alpha = c \underline{\int_a^b} f\, d\alpha \longrightarrow ②$$

$\because$ $f \in R(\alpha)$

$\therefore$ $\overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha$

and from ① & ②, we have

$$\underline{\int_a^b} (cf)\, d\alpha = \overline{\int_a^b} (cf)\, d\alpha = c \int_a^b f\, d\alpha$$

$$\Rightarrow \int_a^b (cf)\, d\alpha = c \int_a^b f\, d\alpha$$

Method-II

(By using Riemann Stieltjes Sum)

Let $h = cf$.

Given a partition P of $[a,b]$, we have

$$S(P,h,\alpha) = \sum_{i=1}^{n} h(t_i)\Delta\alpha_i = c\sum_{i=1}^{n} f(t_i)\Delta\alpha_i$$

Let $\epsilon > 0$ $\qquad = c\, S(P,h,\alpha)$

$\because$ $f \in R(\alpha)$

$\therefore$ There exists a partition P_1 of $[a,b]$ such that if $P^* \supset P_1$, then

$$\left|S(P^*,T,f) - \int_a^b f\,d\alpha\right| < \epsilon/|c|$$

For any choice of point $T = \{t_1, t_2, \dots, t_n\}$ in the component intervals of P^*

Then

$$\left|S(P^*,T,cf) - \int_a^b cf\,d\alpha\right|$$

$$= \left|c\,S(P^*,T,f) - c\int_a^b f\,d\alpha\right|$$

$$= |c|\left|S(P^*,T,f) - \int_a^b f\,d\alpha\right|$$

$$< |c|\,\frac{\epsilon}{|c|} = \epsilon$$

Thus for $\epsilon > 0$ there exists a partition P_1 of $[a,b]$ such that if $P^* \supset P_1$, then

$$\left|S(P^*,T,cf) - c\int_a^b f\,d\alpha\right| < \epsilon$$

$$\Rightarrow cf \in R_\alpha \text{ and } \int_a^b (cf)\,d\alpha = c\int_a^b f\,d\alpha$$

As required.

(b)# Let f_1, f_2, α be bounded on $[a, b]$.
If $f_1, f_2 \in R\alpha[a\ b]$, Then

$$f_1+f_2 \in R\alpha \text{ and } \int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

and if c_1, c_2 are any real numbers, then
$c_1 f_1 + c_2 f_2 \in R(\alpha)$ and

$$\int_a^b (c_1 f_1 + c_2 f_2)\,d\alpha = c_1 \int_o^b f d\alpha + c_2 \int_a^b f_2\, d\alpha$$

<u>Proof</u># $\because$ $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$
$\therefore$ For a given $\epsilon > 0$, $\exists$ partitions P_1 & P_2 such that

$$\left.\begin{array}{l} U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \epsilon \\ \& \\ U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \epsilon \end{array}\right\} \longrightarrow (1)$$

Let $P = P_1 \cup P_2$
Then

$$U(P, f_1, \alpha) \le U(P_1, f_1, \alpha) < L(P_1, f_1, \alpha) + \epsilon$$
$$\le L(P, f_1, \alpha) + \epsilon$$

$$(\because L(P_1, f_1, \alpha) \le L(P, f_1, \alpha)$$
$$\Rightarrow L(P_1, f_1, \alpha) + \epsilon \le L(P, f_1, \alpha) + \epsilon$$

$$\Rightarrow U(P, f_1, \alpha) < L(P, f_1, \alpha) + \epsilon$$

$$\Rightarrow U(P, f_1, \alpha) - L(P, f_1, \alpha) < \epsilon \longrightarrow (2)$$

Similarly

$$U(P, f_2, \alpha) - L(P, f_2, \alpha) < \epsilon \longrightarrow (3)$$

Let $P = \{a = x_0, x_1, x_2 \ldots \ldots x_n = b\}$ and

$f = f_1 + f_2$

Let

$M_i = \operatorname{Sup} f(x)$ on $[x_{i-1}, x_i]$.

$m_i = \operatorname{Inf} f(x)$ on " " ".

$M_i^{(1)} = \operatorname{Sup} f_1(x)$ on " " "

$m_i^{(1)} = \operatorname{Inf} f_1(x)$ on " " "

$M_i^{(2)} = \operatorname{Sup} f_2(x)$ on " " "

$m_i^{(2)} = \operatorname{Inf} f_2(x)$ on " " "

Then

$f_1(x) \leq M_i^{(1)} \quad \forall x \in [x_{i-1}, x_i]$

and $f_2(x) \leq M_i^{(2)}$ " " " "

$\Rightarrow f_1(x) + f_2(x) \leq M_i^{(1)} + M_i^{(2)} \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow f(x) \leq M_i^{(1)} + M_i^{(2)} \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow M_i \leq M_i^{(1)} + M_i^{(2)} \quad \forall x \in [x_{i-1}, x_i]$

$\Rightarrow \sum_{i=1}^{n} M_i \Delta\alpha_i \leq \sum_{i=1}^{n} M_i^{(1)} \Delta\alpha_i + \sum_{i=1}^{n} M_i^{(2)} \Delta\alpha_i$

$$\boxed{U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha)} \rightarrow (4)$$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

Again

$m_i^{(1)} \leq f_1(x) \quad \forall x \in [x_{i-1}, x_i]$

$m_i^{(2)} \leq f_2(x)$ " " " "

$\Rightarrow m_i^{(1)} + m_i^{(2)} \leq f_1(x) + f_2(x)$ " " "

$\Rightarrow m_i^{(1)} + m_i^{(2)} \leq f(x)$ " " "

$$\Rightarrow m_i^{(1)} + m_i^{(2)} \le m_i \quad \forall i$$

$$\Rightarrow \sum_{i=1}^{n} m_i^{(1)} \Delta\alpha_i + \sum_{i=1}^{n} m_i^{(2)} \Delta\alpha_i \le \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$\Rightarrow \boxed{L(P, f_1, \alpha) + L(P, f_2, \alpha) \le L(P, f, \alpha)} \longrightarrow (5)$$

Adding (4) & (5), we have

$$U(P, f, \alpha) + \cancel{L(P, f_1, \alpha)} + L(P, f_2, \alpha)$$
$$\le U(P, f_1, \alpha) + U(P, f_2, \alpha) + \cancel{L(P, f, \alpha)}$$

$$\Rightarrow U(P, f, \alpha) - L(P, f, \alpha)$$
$$\le U(P, f_1, \alpha) - L(P, f_1, \alpha) + U(P, f_2, \alpha) - L(P, f_2, \alpha)$$
$$< \epsilon + \epsilon = 2\epsilon = \epsilon' \quad (\text{By using (2) \& (3)})$$

$$\Rightarrow U(P, f_1+f_2, \alpha) - L(P, f_1+f_2, \alpha) < \epsilon' = 2\epsilon$$

$\because$ ϵ is an arbitrary number and ϵ' depends upon ϵ

$\therefore$ ϵ' is also arbitrary number

$\Rightarrow$ For any arbitrary ϵ', we have found a partition P such that

$$U(P, f_1+f_2, \alpha) - L(P, f_1+f_2, \alpha) < \epsilon'$$

$\Rightarrow$ Riemann's condition of integrability is true

$\Rightarrow$ $f_1 + f_2 \in R(\alpha)$

For the same partition P as above we have

$$U(P, f_j, \alpha) \ge \overline{\int_a^b} f_j \, d\alpha = \int_a^b f_j \, d\alpha \quad \text{for } j = 1, 2$$

$\because f_j \in R(\alpha)$

$$\Rightarrow U(P, f_j; \alpha) \geq \int_a^b f_j \, d\alpha \longrightarrow (6)$$

with the same partition P, we have

$$U(P, f_j, \alpha) < \int_a^b f_j \, d\alpha + \epsilon \longrightarrow (7)$$

(By definition of infimum)

Also for $f = f_1 + f_2$, we have

$$U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha)$$

$$< \int_a^b f_1 d\alpha + \epsilon + \int_a^b f_2 d\alpha + \epsilon \quad \text{By using (7)}$$

$$\Rightarrow \overline{\int_a^b} f d\alpha \leq U(P, f, \alpha) < \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + 2\epsilon$$

$\because$ $f = f_1 + f_2 \in R(\alpha)$ proved above

$$\therefore \overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha = \int_a^b f d\alpha$$

$$\Rightarrow \int_a^b f d\alpha < \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + 2\epsilon \longrightarrow (8)$$

$\because$ ϵ is an arbitrary number which may be very very close to zero

$$\therefore \int_a^b f d\alpha \leq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \longrightarrow (9)$$

Similarly considering the lower sum, we have

$$L(P, f, \alpha) \geq L(P, f_1, \alpha) + L(P, f_2, \alpha)$$

$$L(P, f, \alpha) > \int_a^b f_1 d\alpha - \epsilon + \int_a^b f_2 d\alpha - \epsilon$$

$$\Rightarrow \underline{\int_a^b} f d\alpha \geq L(P, f, \alpha) > \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha - 2\epsilon.$$

$\because$ ϵ is an arbitrary number

$\therefore$ $\int_a^b f\,d\alpha \geq \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$ $\longrightarrow$ (10)

By (9) and (10), we have

$$\int_a^b f\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha.$$

$$\Rightarrow \int_a^b (f_1+f_2)\,d\alpha = \int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha$$ Proved.

Method-II
(Using Riemann Stieltjes Sum)

Let $\epsilon > 0$

$\because$ $f_1 \in R(\alpha)$

$\therefore$ There exists a partition P_1 of $[a.b]$ such that if $P^* \supset P_1$, then

$$\left| S(f_1, P^* T) - \int_a^b f_1\,d\alpha \right| < \epsilon/2 \longrightarrow (1)$$

For any choice of points T in intervals $[x_{i-1}, x_i]$

Similarly there exists a partition P_2 of $[a.b]$ such that if $P^* \supset P_2$, then

$$\left| S(f_2, P^*, T) - \int_a^b f_2\,d\alpha \right| < \epsilon/2 \longrightarrow (2)$$

Let $P = P_1 \cup P_2$. Then $P^* \supset P$

Thus there exists a partition P such that if $P^* \supset P$, then (1) & (2) both holds

So P^* finer than P, we have

$$\left| S(f_1+f_2, P^*, T) - \left(\int_a^b f_1\,d\alpha + \int_a^b f_2\,d\alpha \right) \right|$$

$$= \left| S(f_1, P^*. T) + S(f_2, P^*. T) - \int_a^b f_1\,d\alpha - \int_a^b f_2\,d\alpha \right|$$

$$\leq \left| S(f_1.P^*.T) - \int_a^b f_1 d\alpha \right| + \left| S(P^*.f_2,T) - \int_a^b f_2 d\alpha \right|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon$$

Thus there exists a partition P for $\epsilon > 0$ such that if $P^* \supset P$, then

$$\left| S(f_1+f_2, P^*.T) - \left(\int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \right) \right| < \epsilon$$

$\Rightarrow$ $f_1 + f_2 \in R(\alpha)$ and

$$\int_a^b (f_1+f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \text{ (Proved.)}$$

Proof of $c_1 f_1 + c_2 f_2 \in R(\alpha)$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

Let $h = c_1 f_1 + c_2 f_2$

Given a partition $P = \{a = x_0, x_1, x_2 \cdots x_n = b\}$ we can write for any choice of points t_i in $[x_{i-1}, x_i]$

$$S(P, h, \alpha) = \sum_{i=1}^{n} h(t_i) \Delta\alpha_i$$

$$= \sum_{i=1}^{n} [c_1 f_1(t_i) + c_2 f_2(t_i)] \Delta\alpha_i$$

$$= c_1 \sum f_1(t_i) \Delta\alpha_i + c_2 \sum_{i=1}^{n} f_2(t_i) \Delta\alpha_i$$

$$= c_1 S(P, f_1, \alpha) + c_2 S(P, f_2, \alpha)$$

$\because$ $f_1, f_2 \in R(\alpha)$

$\therefore$ Given $\epsilon > 0$, we choose P'_ϵ so that $P'_\epsilon \subset P$ implies

$$\left| S(P, f_1, \alpha) - \int_a^b f_1 d\alpha \right| < \epsilon \rightarrow (1)$$

Similarly we choose P_ϵ'' so that $P_\epsilon'' \subset P$ implies

$$|S(P, f_2, \alpha) - \int_a^b f_2 \, d\alpha| < \epsilon \longrightarrow (2)$$

If we take $P_\epsilon = P_\epsilon' \cup P_\epsilon''$, then, for P finer than P, we have

$$\left| S(P, h, \alpha) - \left(c_1 \int_a^b f_1 d\alpha + c_2 \int_a^b f_2 d\alpha \right) \right|$$

$$= \left| c_1 S(P, f_1, \alpha) + c_2 S(P, f_2, \alpha) - c_1 \int_a^b f_1 d\alpha - c_2 \int_a^b f_2 d\alpha \right|$$

$$\leq \left| c_1 \left(S(P, f_1, \alpha) - \int_a^b f_1 d\alpha \right) + c_2 \left(S(P, f_2, \alpha) - \int_a^b f_2 d\alpha \right) \right|$$

$$\leq |c_1| \left| S(P, f_1, \alpha) - \int_a^b f_1 d\alpha \right| + |c_2| \left| S(P, f_2, \alpha) - \int_a^b f_2 d\alpha \right|$$

$$\leq |c_1| \epsilon + |c_2| \epsilon = (|c_1| + |c_2|) \epsilon = \epsilon'$$

$\Rightarrow h \in R(\alpha)$ and

$$\int_a^b h \, d\alpha = c_1 \int_a^b f_1 d\alpha + c_2 \int_a^b f_2 d\alpha$$

$$\Rightarrow \int_a^b (c_1 f_1 + c_2 f_2) d\alpha = c_1 \int_a^b f_1 d\alpha + c_2 \int_a^b f_2 d\alpha.$$

More General Case

By induction on k, we can prove the following more general case
If $f_1, f_2, \dots f_k$ are bounded functions on $[a, b]$ and all are Riemann-Stieltjes integrable on $[a, b]$ w.r.t α and if $c_1, c_2, \dots c_k$ are real numbers, then $\sum_{i=1}^{k} c_i f_i \in R_\alpha[a, b]$ and

$$\int_a^b \left[\sum_{i=1}^{k} c_i f_i \right] d\alpha = \sum_{i=1}^{k} c_i \int_a^b f_i \, d\alpha$$

Remarks # The above properties show that the integral operates in a linear fashion on integrand.

2) # Order Preserving Properties

Under the assumption that α is monotone increasing, several useful order preserving properties of the integral can be proved. We prove the following properties.

Theorem # Let the integrator α be bounded and monotone increasing on $[a, b]$. Then.

(a) # $f_1, f_2 \in R(\alpha)$ and

$$f_1(x) \leq f_2(x) \quad \forall x \in [a, b], \text{then}$$

$$\int_a^b f_1 \, d\alpha \leq \int_a^b f_2 \, d\alpha$$

(b) # If $f(x) \geq 0 \quad \forall x \in [a, b]$, Then

$$\int_a^b f \, d\alpha \geq 0$$

(c) # (i) If $f \in R(\alpha)$ on $[a \; b]$, then $|f| \in R(\alpha)$ and

$$\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, d\alpha$$

(ii) $f^2 \in R(\alpha)$ on $[a \; b]$

(d) if $|f(x)| \leq M \quad \forall x \in [a \cdot b]$ and $f \in R_\alpha[a, b]$, Then

$$\left| \int_a^b f \, d\alpha \right| \leq M[\alpha(b) - \alpha(a)]$$

Proof # (a) & (b) first we prove the property (b) and apply it to (b).

Let $f \in R_\alpha[a, b]$ and $f(x) \geq 0 \quad \forall x \in [a, b]$

Let $P = \{a = x_0, x_1, x_2 \ldots \ldots x_n = b\}$ be any partition of $[a \; b]$.

$\because$ $f(x) \geq 0$ $\forall x \in [a, b]$ and α is also monotonically increasing on $[a, b]$

$\therefore$ $0 \leq m_i \leq M_i$ on $[x_{i-1}, x_i]$

It follows that

$$\int_a^b f(x)\,d\alpha \geq L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \geq 0$$

$$\Rightarrow \int_a^b f(x)\,d\alpha(x) \geq 0$$

(a)# Let $f_1, f_2 \in R_\alpha[a, b]$

$\because$ $f_1(x) \leq f_2(x)$ $\forall x \in [a\ b]$

$\therefore$ $(f_2 - f_1)(x) \geq 0$ $\forall x \in [a. b]$

By above result we have

$$\int_a^b (f_2 - f_1)\,d\alpha \geq 0$$

$$\Rightarrow \int_a^b f_2\,d\alpha - \int_a^b f_1\,d\alpha \geq 0$$

$$\Rightarrow \int_a^b f_2\,d\alpha \geq \int_a^b f_1\,d\alpha.$$

$$\Rightarrow \int_a^b f_1\,d\alpha \leq \int_a^b f_2\,d\alpha$$

<u>OR</u>

$\because$ $f_1, f_2 \in R_\alpha[a\ b]$

$\therefore$ $\int_a^b f_1\,d\alpha = \overline{\int_a^b} f_1\,d\alpha = \underline{\int_a^b} f_1\,d\alpha$

and

$\int_a^b f_2\,d\alpha = \overline{\int_a^b} f_2\,d\alpha = \underline{\int_a^b} f_2\,d\alpha$

Again

$\because$ $f_1(x) \leq f_2(x)$ $\forall x \in [a. b]$

$$f_1(x) \le f_2(x) \quad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow m_i^{(1)} \le m_i^{(2)} \quad \forall i$$

$$\Rightarrow \sum_{i=1}^{n} m_i^{(1)} \Delta\alpha_i \le \sum_{i=1}^{n} m_i^{(2)} \Delta\alpha_i \quad \because \Delta\alpha_i \geq 0$$

$$\Rightarrow L(P, f_1, \alpha) \le L(P, f_2, \alpha) \quad \forall P$$

$$\Rightarrow \underline{\int_a^b} f_1 \, d\alpha \le \underline{\int_a^b} f_2 \, d\alpha.$$

$$\Rightarrow \int_a^b f_1 \, d\alpha \le \int_a^b f_2 \, d\alpha.$$

(C) # (i) If $f \in R_\alpha[a, b]$, then $|f| \in R_\alpha[a, b]$.

Let $P = \{a = x_0, x_1, x_2, \ldots\ldots x_n = b\}$ be any partition of $[a, b]$

Let
$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$$
$$m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$$
$$M_i' = \sup\{|f(x)| : x_{i-1} \le x \le x_i\}$$
$$m_i' = \inf\{|f(x)| : x_{i-1} \le x \le x_i\}$$

Since for any real numbers c, d, we have.

$$\big||c| - |d|\big| \le |c - d|$$

Therefore
$$\big||f(x)| - |f(y)|\big| \le |f(x) - f(y)| \quad \forall x, y \in [x_{i-1}, x_i]$$

$$\Rightarrow |M_i' - m_i'| \le |M_i - m_i|$$

$$\Rightarrow M_i' - m_i' \le M_i - m_i \quad \because M_i' \geq m_i' \ \& \ M_i \geq m_i$$

$$\Rightarrow \sum_{i=1}^{n} (M_i' - m_i') \Delta\alpha_i \le \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$$

$$\Rightarrow \sum_{i=1}^{n} M_i' \Delta\alpha_i - \sum_{i=1}^{n} m_i' \Delta\alpha_i \le \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$U(P,|f|,\alpha) - L(P,|f|,\alpha) \leq U(P,f\alpha) - L(P,f\alpha)$

This condition holds for every partition P of $[a,b]$ $\longrightarrow$ ①

$\because$ $f \in R_\alpha[a,b]$

$\therefore$ for $\epsilon > 0$, $\exists$ a partition P_1 of $[a\ b]$ such that

$U(P_1, f, \alpha) - L(P_1, f, \alpha) < \epsilon$ $\longrightarrow$ ②

But by ①, we have.

$U(P_1,|f|\alpha) - L(P_1,|f|\alpha) \leq U(P_1 f\alpha) - L(P_1, f, \alpha) < \epsilon$

$\Rightarrow$ Riemann's condition holds for $|f|$

$\Rightarrow$ $|f| \in R_\alpha[a,b]$

$$\left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha$$

Since $-|f(x)| \leq f(x) \leq |f(x)|$

and for $f_1 \leq f_2$ $\forall x \in [a,b]$

we have $\int_a^b f_1\,d\alpha \leq \int_a^b f_2\,d\alpha$.

Therefore

$$-\int_a^b |f(x)|\,d\alpha \leq \int_a^b f(x)\,d\alpha \leq \int_a^b |f|\,d\alpha$$

$$\Rightarrow \quad \left|\int_a^b f\,d\alpha\right| \leq \int_a^b |f|\,d\alpha.$$

<u>OR</u>

$|f| \in R_\alpha[a,b]$. This can also be proved as under. Define functions f_1 and f_2 as under

$$f_1(x) = \begin{cases} f(x) & f(x) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_2(x) = \begin{cases} -f(x) & f(x) \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then. $f(x) = f_1(x) - f_2(x)$ and $|f(x)| = f_1(x) + f_2(x)$

Explaination # (1) $f(x) = f_1(x) - f_2(x)$.

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

when $f(x) \geq 0$, then $f_1(x) = f(x)$ and $f_2(x) = 0$ and we have.

$f(x) = f_1(x) - f_2(x) = f(x) - 0$

when $f(x) \leq 0$, then $f_2(x) = -f(x)$ and $f_1(x) = 0$ and we have

$f(x) = f_1(x) - f_2(x) = 0 - (-f(x)) = f(x)$

Hence always $f(x) = f_1(x) - f_2(x)$.

(2) $|f(x)| = f_1(x) + f_2(x)|$

when $f(x) \geq 0$ $f_1(x) = f(x)$ and $f_2(x) = 0$

Now $|f(x)| = f_1(x) + f_2(x) = f_1(x) + 0 = f(x)$

when $f(x) \leq 0$ $f_2(x) = -f(x)$ and $f_1(x) = 0$

Now $|f(x)| = -f(x)$ and $f_1(x) + f_2(x) = 0 - f(x)$

$\Rightarrow |f(x)| = f_1(x) + f_2(x)$ always.

$\because f \in R_\alpha[a, b]$

$\therefore$ For $\epsilon > 0$ $\exists$ a partition P of $[a\ b]$ s.t

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

Let $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$

$M_i' = \sup\{f_1(x) : x_{i-1} \leq x \leq x_i\}$

$m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$

$m_i' = \inf\{f_1(x) :$ $x_{i-1} \leq x \leq x_i\}$

Now $f_1 \geq 0$ when $f(x) \geq 0$ i.e $f_1 = f(x)$ when $f \geq 0$

We have

$$\operatorname{Sup}|f| - \operatorname{Inf}|f| \leq \operatorname{Sup} f - \operatorname{Inf} f$$

$$\Rightarrow \operatorname{Sup} f_1 - \operatorname{Inf} f_1 \leq \operatorname{Sup} f - \operatorname{Inf} f \text{ on } [x_{i-1}, x_i]$$

$$\Rightarrow M_i' - m_i' \leq M_i - m_i$$

$$\Rightarrow \sum_{i=1}^{n}(M_i' - m_i')\Delta\alpha_i \leq \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i$$

$$\Rightarrow \sum_{i=1}^{n} M_i'\Delta\alpha_i - \sum m_i'\Delta\alpha_i \leq \sum_{i=1}^{n} M_i\Delta\alpha_i - \sum_{i=1}^{n} m_i\Delta\alpha_i$$

$$U(f_1, P, \alpha) - L(f_1, P, \alpha) \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$$

$$\Rightarrow f_1 \in R_\alpha[a, b]$$

Now $f_2(x) = -f_1(x)$

$\because f_1 \in R_\alpha[a, b]$

$\therefore -f_1 = -1 f_1 \in R_\alpha[a, b]$

$\Rightarrow f_2 \in R_\alpha[a, b]$

$\because$ If $f \in R(\alpha)$ Then $cf \in R(\alpha)$ for $c \in R$

Hence $|f| = f_1 + f_2 \in R_\alpha[a, b]$

Remarks # The converse of the above result is not true i.e if $|f|$ is integrable, then f may or may not be integrable e.g if we take

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$$

and $\alpha(x) = x$

Then

$$\overline{\int_a^b} f\,d\alpha = b - a \qquad \underline{\int_a^b} f\,d\alpha = -(b - a)$$

So that f is not integrable.

But since $|f(x)| = 1 \quad \forall x$, Therefore

$$\int_a^b |f(x)|\,d\alpha = b - a \quad \text{i.e } |f| \text{ is integrable.}$$

If $f \in R(\alpha)$, then $f^2 \in R(\alpha)$

We have

$$\operatorname{Sup} f^2 = [\operatorname{Sup}|f|]^2$$

and

$$\operatorname{Inf} f^2 = [\operatorname{Inf}|f|]^2$$

$\because$ $f \in R(\alpha)[a, b]$

$\therefore$ $|f| \in R(\alpha)$

$\Rightarrow |f|$ is bounded

$\Rightarrow \exists$ a number M such that

$$||f|| \leq M$$

$$\Rightarrow |f| \leq M$$

$\because$ $|f| \in R(\alpha)$

$\therefore$ for any $\epsilon > 0$ $\exists$ a partition P such that

$$U(P, |f|, \alpha) - L(P, |f|, \alpha) < \epsilon/_{2M} \longrightarrow ①$$

Let $M_i' = \operatorname{Sup}|f(x)| \quad \forall x \in [x_{i-1}, x_i]$

$m_i' = \operatorname{Inf}|f(x)| \quad \forall x \in [x_{i-1}, x_i]$

Then $M_i'^2 = \operatorname{Sup} f^2$ on $[x_{i-1}, x_i]$

$m_i'^2 = \operatorname{Inf} f^2$ on $[x_{i-1}, x_i]$

$$\because \left.\begin{array}{l} M_i' \leq M \\ \& \ m_i' \leq M \end{array}\right\} \Rightarrow M_i' + m_i' \leq 2M \longrightarrow ②$$

$$U(P, f^2, \alpha) - L(P, f^2, \alpha) = \sum_{i=1}^{n} (M_i'^2 - m_i'^2) \Delta\alpha_i$$

$$= \sum_{i=1}^{n} (M_i' + m_i')(M_i' - m_i') \Delta\alpha_i$$

$$U(P, f^2, \alpha) - L(P, f^2, \alpha) \leq 2M \sum_{i=1}^{n} (M_i' - m_i')\Delta\alpha_i \quad \because M_i' + m_i' \leq 2M$$

$$= 2M [U(P, |f|, \alpha) - L(P, |f|, \alpha)]$$

$$\leq 2M \cdot \frac{\epsilon}{2M} = \epsilon \quad \text{by } (1)$$

$\Rightarrow f^2 \in R_\alpha[a,b]$

Remarks # (1) If f is an arbitrary function, then

$$f^2 = |f|^2$$

$$\therefore \sup f^2 = [\sup |f|]^2$$

Also we have prove that
if $f \in R(\alpha)$ on $[a,b]$, then $f^2 \in R(\alpha)[a,b]$
if $f \in R_\alpha[a,b]$, then $|f| \in R_\alpha[a,b]$

Consequently $|f|^2 \in R_\alpha[a\ b]$

3) # Algebraic Properties of Integral

Under the assumption that α is monotone increasing, several algebraic properties of the integral can be derived.

Theorem # Suppose that the integrator α is bounded and monotone increasing on $[a,b]$ and f, g are in $R_\alpha[a,b]$, then

(a) # $f^2 \in R_\alpha[a\ b]$

(b) # The function fg is also in $R_\alpha[a,b]$

(c) # If there exist m and M such that $0 < m \leq |f| \leq M$, Then $1/f \in R_\alpha[a\ b]$.

Proof # We have already proved part (a). We prove

here part (b) and c

(b) # $\because f \in R_\alpha[a, b]$, $g \in R_\alpha[a, b]$

$\Rightarrow f+g \in R_\alpha[a, b]$ & $f-g \in R_\alpha[a, b]$

Again if $f \in R_\alpha[a, b]$, then $f^2 \in R_\alpha[a, b]$

Therefore $(f+g)^2 \in R_\alpha[a, b]$ & $(f-g)^2 \in R_\alpha[a, b]$

$\Rightarrow (f+g)^2 - (f-g)^2 \in R_\alpha[a, b]$

$\Rightarrow \frac{1}{4}[(f+g)^2 - (f-g)^2] \in R_\alpha[a, b]$ $\because$ if $f \in R(\alpha)$ then $cf \in R(\alpha)$ for any real c

$\Rightarrow fg \in R_\alpha[a, b]$

Direct Method

$\because f, g \in R_\alpha[a, b]$

$\therefore f, g$ are bounded on $[a, b]$

$\Rightarrow \exists$ numbers k_1, k_2 such that

$|f(x)| \le k_1 \quad \forall x \in [a, b]$

and $|g(x)| \le k_2 \quad \forall x \in [a, b]$

Let $k = \max\{k_1, k_2\}$

Then $|f(x)| \le k \quad \forall x \in [a, b]$

and $|g(x)| \le k \quad \forall x \in [a, b]$

Let $P = \{a = x_0, x_1, x_2 \ldots\ldots x_n = b\}$ be any partition of $[a, b]$ and let

$M_i = \sup(fg)$ on $[x_{i-1}, x_i]$

$m_i = \operatorname{Inf}(fg)$ on " "

$M_i' = \sup f$ on " "

$m_i' = \operatorname{Inf} f$ on " "

$M_i'' = \sup g$ on " "

$m_i'' = \operatorname{Inf} g$ on " "

We have $\forall\ x, y \in [x_{i-1}, x_i]$

$$f(y)g(y) - g(x)f(x) = f(y)g(y) - g(x)f(x) + f(x)g(y) - f(x)g(y)$$

$$= g(y)[f(y) - f(x)] + f(x)[g(y) - g(x)]$$

$$\Rightarrow |f(y)g(y) - g(x)f(x)| \le |g(y)|\,|f(y) - f(x)| + |f(x)|\,|g(y) - g(x)|$$

$$(M_i - m_i) \le k(M_i' - m_i') + k(M_i'' - m_i'') \longrightarrow ①$$

This condition holds for all partitions of $[a, b]$

$\because\ f, g \in R_\alpha[a, b]$

$\therefore$ For a given $\epsilon > 0$, $\exists$ partitions P_1, P_2 such that

$$U(P_1, f, \alpha) - L(P_1, f, \alpha) < \frac{\epsilon}{2k} \longrightarrow ②$$

and

$$U(P_2, g, \alpha) - L(P_2, g, \alpha) < \frac{\epsilon}{2k} \longrightarrow ③$$

Let $P^* = P_1 \cup P_2$

Then

$$U(P^*, f, \alpha) \le U(P_1, f, \alpha) \longrightarrow ④$$

$$L(P^*, f, \alpha) \ge L(P_1, f, \alpha)$$

$$\Rightarrow -L(P^*, f, \alpha) \le -L(P_1, f, \alpha) \longrightarrow ⑤$$

Adding ④ & ⑤

$$U(P^*, f, \alpha) - L(P^*, f, \alpha) \le U(P_1, f, \alpha) - L(P_1, f, \alpha) < \epsilon/2k$$

$$\Rightarrow U(P^*, f, \alpha) - L(P^*, f, \alpha) < \frac{\epsilon}{2k} \longrightarrow ⑥$$

Similarly

$$U(P^*, g, \alpha) - L(P^*, g, \alpha) < \frac{\epsilon}{2k} \longrightarrow ⑦$$

Now for the same partition P^* Condition ① holds and we have

$$U(P^*, fg, \alpha) - L(P^*, fg, \alpha) = \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i$$

$$\leq \sum_{i=1}^{n} [k(M_i' - m_i') + k(M_i'' - m_i'')]\Delta\alpha_i$$

$$= k[\Sigma M_i' \Delta\alpha_i - \Sigma m_i' \Delta\alpha_i] + k[\Sigma M_i'' \Delta\alpha_i - \Sigma m_i'' \Delta\alpha_i]$$

$$= k[U(P^*, f, \alpha) - L(P^*, f, \alpha)] + k[U(P^*, g, \alpha) - L(P^*, g, \alpha)]$$

$$< k \cdot \left(\frac{\epsilon}{2k}\right) + k\left(\frac{\epsilon}{2k}\right) \quad \text{using ⑥ \& ⑦}$$

$$= \epsilon$$

$$\Rightarrow U(P^*, fg, \alpha) - L(P^*, fg, \alpha) < \epsilon \quad \longrightarrow ⑧$$

So for an arbitrary $\epsilon > 0$, we have found a partition P^* of $[a, b]$ such that ⑧ holds

$\Rightarrow$ Riemann's Condition is True

$\Rightarrow fg \in R_\alpha[a, b]$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

(C) # $\because$ given that

$$|f(x)| \geq m \quad \forall x \in [a, b]$$

$$\therefore \frac{1}{|f(x)|} \leq \frac{1}{m}$$

$$\Rightarrow \frac{1}{f(x)} \text{ is bounded}$$

Let $P_0 = \{a = x_0, x_1, \text{-----} \; x_n = b\}$ be a partition of $[a, b]$

Let $M_i' = \sup\left(\frac{1}{f(x)}\right)$ on $[x_{i-1}, x_i]$

$m_i' = \inf\left(\frac{1}{f(x)}\right)$ on $[x_{i-1}, x_i]$

$M_i = \sup f(x)$ on $[x_{i-1}, x_i]$

$m_i = \inf f(x)$ on " "

Now $\forall\, x, y \in [x_{i-1}, x_i]$, we have

$$\left|\frac{1}{f(y)} - \frac{1}{f(x)}\right| = \left|\frac{f(y) - f(x)}{f(x)\, f(y)}\right|$$

$$= \frac{|f(y) - f(x)|}{|f(x)|\,|f(y)|}$$

$$M_i' - m_i' \leq \frac{M_i - m_i}{m^2} \quad \forall \text{ partitions} \rightarrow (1)$$

$\because\ f \in R_\alpha[a, b]$

$\therefore$ For $\epsilon > 0$ $\exists$ a partition P of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon m^2$$

$$\Rightarrow \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i < \epsilon m^2 \rightarrow (2)$$

Now for the same partition P, we have

$$M_i' - m_i' \leq \frac{M_i - m_i}{m^2}$$

$$\Rightarrow \sum_{i=1}^{n} (M_i' - m_i')\Delta\alpha_i \leq \frac{1}{m^2} \sum (M_i - m_i)\Delta\alpha_i$$

$$\Rightarrow U\left(P, \frac{1}{f}, \alpha\right) - L\left(P, \frac{1}{f}, \alpha\right) < \frac{1}{m^2} m^2 \epsilon = \epsilon \text{ using } (2)$$

$\Rightarrow$ Riemann's Condition is true

$\Rightarrow \frac{1}{f} \in R_\alpha[a, b]$

Remarks # The above Theorem shows that $R_\alpha[a, b]$ is a Commutative Ring with multiplicative idenity. Part(c) identifies units in $R_\alpha[a, b]$

Problem # (Related to part (c) of above theorem)
Let α be an increasing function on $[a, b]$. Let $f \in R_\alpha[a, b]$ and suppose that for some +ve number M, $|f(x)| \geq M$ $\forall x \in [a, b]$. Prove that $1/f \in R_\alpha[a, b]$

Problem # Prove that if f is a continuous non-negative function on $[a, b]$ and

$$\int_a^b f(x)\,dx = 0, \text{ then } \forall x \in [a, b],\ f(x) = 0$$

Then $f(x) = 0$ $\forall x \in [a, b]$

Sol # $\because$ f is non-negative

$\therefore$ $f(x) \geq 0$ $\forall x \in [a, b]$

$\Rightarrow$ for any partition P of $[a, b]$

$0 \leq M_i = \sup f(x)$ $\forall x \in [x_{i-1}, x_i]$

$0 \leq m_i = \inf f(x)$ $\forall x \in$ " "

But $\int_a^b f(x)\,dx = 0$

$\Rightarrow$ $\sup_P L(P, f, \alpha) = 0$

$\Rightarrow$ $L(P, f, \alpha) = 0$ $\forall$ partition P of $[a, b]$

$\Rightarrow$ $\sum_{i=1}^{n} m_i \Delta x_i = 0$ $\forall P$ of $[a, b]$

$\because$ It is sum of non-negative quantities

$\therefore$ $m_i \Delta x_i = 0$ $\forall i$ and $\forall P$

$\Rightarrow$ $m_i = 0$ $\forall i$ $\because \Delta x_i \neq 0$ $\forall i$

$\Rightarrow$ $f(x) = 0$ $\forall x \in [a, b]$

(Proved)

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* Stay at the crossing of struggle, sincerity and motivation you will see your destination nearer to you. //

4) # **Theorem #** Let f, α_1, α_2 be bounded functions on $[a\ b]$ and c be a +ve constant.

a) # If $f \in R(\alpha_1)$ & $f \in R(\alpha_2)$ i.e

$$f \in R_{\alpha_1}[a,b] \cap R_{\alpha_2}[a,b], \text{ then}$$

$$f \in R_{\alpha_1+\alpha_2}[a\ b]$$

b) # If $f \in R_{\alpha_1}[a,b]$, then $f \in R_{c\alpha_1}[a,b]$

Proof # a) $\because$ $f \in R_{\alpha_1}[a\ b]$ & $f \in R_{\alpha_2}[a.b]$

$\therefore$ For $\epsilon > 0$, $\exists$ partitions P_1 & P_2 such that

$$U(P_1, f\ \alpha_1) - L(P_1, f\ \alpha_1) < \epsilon/2 \longrightarrow (1)$$

and

$$U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) < \epsilon/2 \longrightarrow (2)$$

Set $P = P_1 \cup P_2$

Then

$$\left.\begin{aligned} U(P, f\ \alpha_1) - L(P, f, \alpha_1) < \epsilon/2 \\ U(P, f\ \alpha_2) - L(P, f\ \alpha_2) < \epsilon/2 \end{aligned}\right\} \longrightarrow (3)$$

Let $M_i = \sup f(x)$ on $[x_{i-1}\ x_i]$

Then

$$\Delta(\alpha_1+\alpha_2)_i = (\alpha_1+\alpha_2)(x_i) - (\alpha_1+\alpha_2)(x_{i-1})$$

$$= \alpha_1(x_i) - \alpha_1(x_{i-1}) + \alpha_2(x_i) - \alpha_2(x_{i-1})$$

$$= \Delta\alpha_{1i} + \Delta\alpha_{2i}$$

$$U(P, f, \alpha_1+\alpha_2) = \sum_{i=1}^{n} M_i \Delta(\alpha_1+\alpha_2)_i$$

$$= \sum_{i=1}^{n} M_i [\Delta\alpha_{1i} + \Delta\alpha_{2i}]$$

$$= \sum_{i=1}^{n} M_i \Delta\alpha_{1i} + \sum_{i=1}^{n} M_i \Delta\alpha_{2i}$$

$$= U(P, f, \alpha_1) + U(P, f, \alpha_2)$$

Similarly

$$L(P, f, \alpha_1+\alpha_2) = L(P, f, \alpha_1) + L(P, f, \alpha_2)$$

$$U(P, f, \alpha_1+\alpha_2) - L(P, f, \alpha_1+\alpha_2)$$

$$= U(P, f, \alpha_1) + U(P, f, \alpha_2) - [L(P, f, \alpha_1) + L(P, f, \alpha_2)]$$

$$= U(P, f, \alpha_1) - L(P, f, \alpha) + U(P, f, \alpha_2) - L(P, f, \alpha_2)$$

$$< \epsilon/2 + \epsilon/2 = \epsilon \quad \text{By using } ③$$

$\Rightarrow$ Riemann's Condition is satisfied.

$\Rightarrow \quad f \in R_{\alpha_1+\alpha_2}[a, b]$

Again

Since

$$U(P, f, \alpha_1+\alpha_2) = U(P, f, \alpha_1) + U(P, f, \alpha_1)$$

$$\Rightarrow \inf_P U(P, f, \alpha_1+\alpha_2) = \overline{\int_a^b} f \, d(\alpha_1+\alpha_2)$$

$$\leq U(P, f, \alpha_1+\alpha_2)$$

$$= U(P, f, \alpha_1) + U(P, f, \alpha_2)$$

$$\Rightarrow \inf_P U(P, f, \alpha_1+\alpha_2) \leq \inf_P U(P, f, \alpha_1) + \inf U(P, f, \alpha_2)$$

$$\Rightarrow \int_a^b f \, d(\alpha_1+\alpha_2) \leq \int_a^b f \, d\alpha_1 + \int_a^b f \, d\alpha_2 \quad \because f \in R(\alpha_1),\ f \in R(\alpha_2),\ f \in R(\alpha_1+\alpha_2)$$

$\longrightarrow ④$

$$\sup_P L(P, f, \alpha_1+\alpha_2) \geq \sup_P L(P, f, \alpha_1) + \sup L(P, f, \alpha_2)$$

$$\Rightarrow \int_a^b f\, d(\alpha_1+\alpha_2) \geq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \longrightarrow (5)$$

from (4) and (5), we have.

$$\int_a^b f\, d(\alpha_1+\alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

Method-II (using Riemann-Stieltjes Sum)

Given any partition P of $[a,b]$, we have

$$S(P, f, \alpha_1+\alpha_2) = \sum_{i=1}^{n} f(t_i)\, \Delta(\alpha_1+\alpha_2)_i$$

$$= \sum_{i=1}^{n} f(t_i)\, \Delta\alpha_{1i} + \sum_{i=1}^{n} f(t_i)\, \Delta\alpha_{2i}$$

$$= S(P, f\, \alpha_1) + S(P, f, \alpha_2)$$

$\because$ $f \in R_{\alpha_1}[a,b]$ and $f \in R_{\alpha_2}[a,b]$

Given $\epsilon > 0$, $\exists$ partitions P_ϵ', P_ϵ'' such that $P_\epsilon' \subset P$ and $P_\epsilon'' \subset P$ implies that

$$\left| S(P, f, \alpha_1) - \int_a^b f d\alpha_1 \right| < \epsilon/2 \longrightarrow (1)$$

and

$$\left| S(P, f\, \alpha_2) - \int_a^b f d\alpha_2 \right| < \epsilon/2 \longrightarrow (2)$$

Let $P_\epsilon = P_\epsilon' \cup P_\epsilon''$. Then for P finer than P_ϵ, we have

$$\left| S(P, f, \alpha_1+\alpha_2) - \left(\int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \right) \right|$$

$$= \left| S(P, f, \alpha_1) + S(P, f, \alpha_2) - \int_a^b f d\alpha_1 - \int_a^b f d\alpha_2 \right|$$

$$\leq \left| S(P, f, \alpha) - \int_a^b f d\alpha_1 \right| + \left| S(P, f, \alpha) - \int_a^b f d\alpha_2 \right|$$

$< \epsilon/2 + \epsilon/2 = \epsilon$

Thus we have traced a partition P_ϵ such that for any P finer than P_ϵ, we have

$$\left| S(P, f, \alpha_1+\alpha_2) - \left(\int_a^b f d\alpha_1 + \int_a^b f d\alpha_2\right) \right| < \epsilon$$

$$\Rightarrow \int_a^b f\, d(\alpha_1+\alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

so $f \in R_{\alpha_1+\alpha_2}[a, b]$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

(b) # $\because f \in R(\alpha)$

$\therefore$ for $\epsilon > 0$, $\exists$ a partition P, such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon/c \longrightarrow (a)$$

For the same partition P, we have.

$$U(P, f, c\alpha) = \sum_{i=1}^{n} M_i \Delta (c\alpha)_i$$

$$= c \sum_{i=1}^{n} M_i \Delta \alpha_i$$

$$= c\, U(P, f, \alpha) \longrightarrow (1)$$

Similarly

$$L(P, f, c\alpha) = c\, L(P, f, \alpha) \longrightarrow (2)$$

$$U(P, f, c\alpha) - L(P, f, c\alpha) = c\, U(P, f, \alpha) - c\, L(P, f, \alpha)$$

$$= c[U(P, f, \alpha) - L(P, f, \alpha)]$$

$$< c \cdot \epsilon/c = \epsilon \quad \text{using (a)}$$

$\Rightarrow$ Riemann Condition is true

$\Rightarrow f \in R(c\alpha)$ on $[a\ b]$

Now

Since $U(P, f, c\alpha) = c\,U(P, f, \alpha) \quad \forall P$

Therefore

$$\inf_P U(P, f, c\alpha) = c \inf U(P, f\alpha)$$

$$\overline{\int_a^b} f\,d(c\alpha) = c\,\overline{\int_a^b} f\,d\alpha$$

But $f \in R(\alpha)$ & $f \in R(c\alpha)$

$$\therefore \quad \overline{\int_a^b} f\,d(c\alpha) = \underline{\int_a^b} f\,d(c\alpha) = \int_a^b f\,d(c\alpha)$$

and

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha$$

$$\Rightarrow \quad \int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha$$

Similarly we have

$$L(P, f, c\alpha) = c\,L(P, f, \alpha) \quad \forall P$$

$$\Rightarrow \quad \sup_P L(P, f\,c\alpha) = c \sup_P L(P, f, \alpha)$$

$$\underline{\int_a^b} f\,d(c\alpha) = c\,\underline{\int_a^b} f\,d\alpha$$

$\because f \in R(\alpha)$ and $f \in R(c\alpha)$

Therefore

$$\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha$$

Remarks # Part (a) of above Theorem can actually be proved as. If $f \in R_{\alpha_1}[a, b]$, $f \in R_{\alpha_2}[a, b]$, c_1, c_2 are any constants, then $f \in R_{c_1\alpha_1 + c_2\alpha_2}[a\ b]$ and $\int_a^b f\,d(c_1\alpha_1 + c_2\alpha_2) = c_1\int_a^b f\,d\alpha_1 + c_2\int_a^b f\,d\alpha_2$

Part(a) of above Theorem shows that the integral operates in a linear fashion on the integrator.

* Part (b) of above theorem actually holds for any constant c not necessarily +ve. This will be proved latter on

Theorem # Let α be monotone increasing and bounded on $[a\ b]$. If $f \in R_\alpha[a,b]$, $a < c < b$, then $f \in R_\alpha[a\ c]$ and $f \in R_\alpha[c,b]$ and

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

Also for any sub-interval $[c\ d]$ of $[a,b]$ $f \in R_\alpha[c\ d]$

Proof # Let $f \in R(\alpha)$ on $[a, b]$

First we prove that

$f \in R(\alpha)$ on $[a, c]$

and $f \in R(\alpha)$ on $[c\ b]$ $\because f \in R(\alpha)$ on $[a,b]$

$\therefore$ for $\epsilon > 0$, $\exists$ a partition P s.that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon \longrightarrow ①$

Let $P = \{a = x_0, x_1, x_2, x_3 \text{-----} x_n = b\}$ be a partition of $[a, b]$ and $P' = \{x_0, x_1, x_2 \text{---} x_k, c, x_{k+1}, \text{----} x_n\}$ be its refinement.

Let $P_1 = \{x_0, x_1, x_2 \text{-----} x_k, c\} = P' \cap [a\ c]$

$P_2 = \{c, x_{k+1}, x_{k+2}, \text{-------} , x_n\} = P' \cap [c, b]$

The P_1 and P_2 are partions of $[a\ c]$ and $[c, b]$ such that

$$P' = P_1 \cup P_2$$

$\because$ Riemann Condition ① holds for any refinement

of P, therefore we have

$$U(P',f,\alpha)-L(P',f,\alpha)<\epsilon \quad \because P'\supset P \longrightarrow (2)$$

Let $M_i = \sup f(x)$ on $[x_{i-1}, x_i]$

and $m_i = \inf f(x)$ on $[x_{i-1}\ x_i]$

$M_{r_1} = \sup f(x)$ on $[x_k\ \ c]$

$m_{r_1} = \inf f(x)$ on $[x_k\ \ c]$

$M_{r_2} = \sup f(x)$ on $[c\ \ x_{k+1}]$

$m_{r_2} = \inf f(x)$ on $[c\ \ x_{k+1}]$

Now

$$U(P',f,\alpha) = \sum_{i=1}^{k} M_i\Delta\alpha_i + M_{r_1}(\alpha(c)-\alpha(x_k)) + M_{r_2}(\alpha(x_{k+1})-\alpha(c))$$

$$+ \sum_{i=k+2}^{n} M_i\Delta\alpha_i$$

$$= \sum_{i=1}^{k} M_i\Delta\alpha_i + M_{r_1}[\alpha(c)-\alpha(x_k)]$$

$$+ M_{r_2}[\alpha(x_{k+1})-\alpha(c)] + \sum_{i=k+2}^{n} M_i\Delta\alpha_i$$

$$= U(P_1,f,\alpha)+U(P_2,f,\alpha) \longrightarrow (3)$$

Also

$$L(P',f,\alpha) = L(P_1,f,\alpha)+L(P_2,f,\alpha) \longrightarrow (4)$$

Subtracting (4) from (3)

$$[U(P_1,f,\alpha)+U(P_2,f,\alpha)]-[L(P_1,f,\alpha)+L(P_2,f,\alpha)]$$

$$= U(P',f,\alpha)-L(P',f,\alpha) < \epsilon \quad \text{by (2)}$$

$$\Rightarrow [U(P_1,f,\alpha)-L(P_1,f,\alpha)]+[U(P_2,f,\alpha)-L(P_2,f,\alpha)]<\epsilon \longrightarrow (4)$$

$\because$ $U(P_1,f,\alpha)-L(P_1,f,\alpha)$ is non-negative

and $U(P_2,f,\alpha)-L(P_2,f,\alpha)$ is non negative

Therefore (4) $\Rightarrow$

$U(P_1, f, \alpha) - L(P_1, f, \alpha) < \epsilon \longrightarrow (5)$

and

$U(P_2, f, \alpha) - L(P_2, f, \alpha) < \epsilon \longrightarrow (6)$

$\Rightarrow f \in R_\alpha[a\ c]$ and $f \in R_\alpha[c\ b]$

OR

We can prove (5) and (6) as under

$$U(P_1, f, \alpha) - L(P_1, f, \alpha) = \sum_{i=1}^{k} (M_i - m_i)\Delta\alpha_i + (M_{r_1} - m_{r_1})[\alpha(c) - \alpha(x_k)]$$

$$< \sum_{i=1}^{k} (M_i - m_i)\Delta\alpha_i + (M_{r_1} - m_{r_1})[\alpha(c) - \alpha(x_k)] + (M_{r_2} - m_{r_2})[\alpha(x_{k+1}) - \alpha(c)] + \sum_{i=k+2}^{n} (M_i - m_i)\Delta\alpha_i$$

$$= U(P', f, \alpha) - L(P', f, \alpha) < \epsilon.$$

$\Rightarrow U(P_1, f, \alpha) - L(P_1, f, \alpha) < \epsilon$

$\Rightarrow f \in R_\alpha[a\ c]$

Similarly $U(P_2, f, \alpha) - L(P_2, f, \alpha) < \epsilon$

$\Rightarrow f \in R_\alpha[c\ b]$

Proof of $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$

$$U(P, f, \alpha) \geq U(P', f, \alpha) = U(P_1, f, \alpha) + U(P_2, f, \alpha)$$

$$\geq \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha$$

$$= \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \quad \because f \in R_\alpha[a c] \ \& \ f \in R_\alpha[c, b]$$

and this condition will hold for any P because for any given P we form refinement P' by inserting point c in between some component interval of P. Therefore

$$\int_a^b f\,d\alpha \geq \int_a^c f\,d\alpha + \int_c^b f\,d\alpha \longrightarrow (7)$$

Again

$$L(P,f,\alpha) \leq L(P',f,\alpha) = L(P_1,f,\alpha) + L(P_2,f,\alpha)$$

$$\leq \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha$$

$$= \int_a^c f\,d\alpha + \int_b^c f\,d\alpha$$

As above this will also hold for all partitions and hence, we have

$$\int_a^b f\,d\alpha = \underline{\int_a^b} f\,d\alpha \leq \int_a^c f\,d\alpha + \int_b^c f\,d\alpha$$

$$\Rightarrow \int_a^b f\,d\alpha \leq \int_a^c f\,d\alpha + \int_b^c f\,d\alpha \longrightarrow (8)$$

By (7) and (8), we have

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

Method-II

Both the above result of Theorem can also be proved as under.

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$.

make P' as above and P_1, P_2 as given above.

$\because f \in R_\alpha[a\ b]$

$\therefore \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha.$

$$U(P,f,\alpha) \geq U(P',f,\alpha) = U(P_1,f,\alpha) + U(P_2,f,\alpha)$$

$$U(P,f,\alpha) \geq U(P_1,f,\alpha)+U(P_2,f,\alpha)$$
$$\geq \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha.$$

and this condition remains true for all partitions of $[a,b]$. Hence

$$\int_a^b f\,d\alpha = \overline{\int_a^b} f\,d\alpha \geq \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha \longrightarrow (a)$$

Again as above

$$L(P,f,\alpha) \leq \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha.$$

$$\Rightarrow \int_a^b f\,d\alpha = \underline{\int_a^b} f\,d\alpha \leq \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha \longrightarrow (b)$$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

Therefore, we have

$$\int_a^b f\,d\alpha \geq \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha \geq \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha$$
$$\geq \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha \geq \int_a^b f\,d\alpha$$

$$\Rightarrow \int_a^b f\,d\alpha = \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha.$$

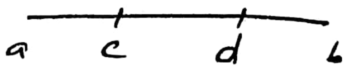
and $$\int_a^b f\,d\alpha = \overline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha.$$

$$\Rightarrow \underline{\int_c^b} f\,d\alpha = \overline{\int_c^b} f\,d\alpha \Rightarrow f \in R_\alpha[b,c]$$

Again $$\int_a^b f\,d\alpha = \overline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha$$
and $$\int_a^b f\,d\alpha = \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha$$
$$\Rightarrow \overline{\int_a^c} f\,d\alpha = \underline{\int_a^c} f\,d\alpha \Rightarrow f \in R_\alpha[a\ c]$$

Further
$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$
Proved.

Let $[c, d]$ be a closed sub-interval of $[a, b]$.
The by above result
$f \in R_\alpha[a, b]$ and $a < d < b$



$\Rightarrow$ $f \in R_\alpha[a, d]$ & $f \in R_\alpha[d, b]$

Again $f \in R_\alpha[a, d]$ and $a < c < d$.

Hence by above result $f \in R_\alpha[c\ d]$ as required.

Remarks #(1) If $a < b$, we define $\int_a^b f\,d\alpha = -\int_b^a f\,d\alpha$ whenever $\int_a^b f\,d\alpha$ exists. We also define $\int_a^a f\,d\alpha = 0$

So $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$

$$\Rightarrow \int_a^b f\,d\alpha + \int_c^a f\,d\alpha + \int_b^c f\,d\alpha = 0$$

(2)# The above Theorem shows that integral is « additive » w.r.t the interval of integration.

(3)# Using mathematical induction we can prove a similar result for a decomposition of $[a, b]$ into a finite number of sub-intervals

4)# The fact that if $f \in R_\alpha[a, b]$, then we can integrate f over any sub-interval $[c, d]$ is an essential tool for the study of the indefinite integral $F(x) = \int_a^x f(t)\,d(t)$ as we will see latter on.

Lemma# If M & m are supremum and infimum of f and M' & m' are supremum and infimum of $|f|$ on $[a, b]$. Then

$$M' - m' \leq M - m$$

Proof# $\because$ For any two real numbers c, d, we have

$$\big||c| - |d|\big| \leq |c - d|$$

Therefore for any $x', x'' \in [a, b]$, we have

$$\big||f(x')| - |f(x'')|\big| \leq |f(x') - f(x'')| \longrightarrow (A)$$

$\because$ $M = \sup f(x)$ on $[a, b]$

and $m = \inf f(x)$ on $[a, b]$

Therefore

$f(x) \leq M \quad \forall x \in [a, b]$

and $f(x) \geq m \quad \forall x \in [a, b]$

$\because$ $x', x'' \in [a, b]$

$\therefore$ $f(x') \leq M \rightarrow (1)$ & $f(x'') \geq m$

$\Rightarrow -m \geq -f(x'')$

or $-f(x'') \leq -m \rightarrow (2)$

Adding (1) & (2)

$$f(x') - f(x'') \leq M - m \longrightarrow (3)$$

By interchanging x' & x''

$$f(x'') - f(x') \leq M - m$$

$$\Rightarrow -[f(x') - f(x'')] \leq M - m \longrightarrow (4)$$

By (3) & (4), we have

$$|f(x') - f(x'')| \leq M - m \longrightarrow (5)$$

Muhammad Hussain
Lecturer (Maths)
Govt. College Asghar Mall
Rawalpindi.

Using ⑤ in Ⓐ, we have

$$\big|\,|f(x')| - |f(x'')|\,\big| \leq M - m \longrightarrow ⑥$$

Given that $M' = \text{Sup}\,|f(x)|$ on $[a, b]$
and $m' = \text{Inf}\,|f(x)|$ on $[a, b]$

$\Rightarrow |f(x)| \leq M'$ & $|f(x)| \geq m'$ $\forall x \in [a, b]$

For any $\epsilon > 0$, we can find two points $x_1, x_2 \in [a, b]$ such that

$$|f(x_1)| > M' - \epsilon$$

$$M' - \epsilon < |f(x_1)| \longrightarrow ⑦$$

and $|f(x_2)| < m' + \epsilon$

$$\Rightarrow -m' + |f(x_2)| < \epsilon \longrightarrow ⑧$$

Adding ⑦ and ⑧

$$M' - m' - \epsilon < |f(x_1)| - |f(x_2)| + \epsilon$$

$$\Rightarrow M' - m' < |f(x_1)| - |f(x_2)| + 2\epsilon \longrightarrow ⑨$$

Now for every $\epsilon > 0$ we can find two points such that ⑨ holds.
Therefore, we have

$$M' - m' < |f(x_1)| - |f(x_2)| \quad \forall x_1, x_2 \in [a, b]$$

$$\Rightarrow M' - m' < |f(x')| - |f(x'')| \longrightarrow ⑩$$

Interchanging x' & x''

$$M' - m' \leq -\big[|f(x')| - |f(x'')|\big] \longrightarrow ⑪$$

By ⑩ and ⑪, we have

$$M' - m' \leq \big| |f(x')| - |f(x'')| \big| \rightarrow (12)$$

By (6) and (12), we have

$$M' - m' \leq \big| |f(x')| - |f(x'')| \big| \leq M - m$$

$$\Rightarrow M' - m' \leq M - m \quad \text{(Proved)}$$

Theorem

Let f be a bounded function on $[a, b]$ and let α be monotone increasing on $[a, b]$. Suppose that $m \leq f(x) \leq M$ for x in $[a, b]$. If $f \in R_\alpha[a, b]$ and g is continuous on $[m, M]$, $g o f \in R_\alpha[a, b]$ OR

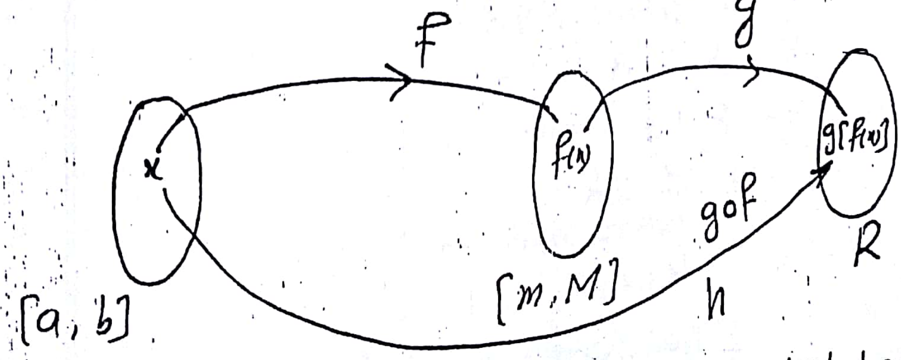
Let $f \in R_\alpha[a, b]$; $m \leq f(x) \leq M$, g be continuous on $[m, M]$ and $h(x) = g[f(x)] = g o f$ on $[a, b]$. Then $h \in R_\alpha[a, b]$

Proof# Let $h = g o f$

$\because$ A function continuous on a closed interval is bounded and g is continuous on $[m, M]$

$\therefore$ g is bounded

Let $|g(t)| < K \quad \forall t \in [m, M]$



Since g is continuous on $[m, M]$ which being closed and bounded is compact, therefore g is uniformly continuous

on [~~a~~, M] because a function continuous on a compact set is uniformly continuous on that set.

Therefore for $\epsilon > 0$, there exists δ, $0 < \delta < \dfrac{\epsilon}{2K + \alpha(b) - \alpha(a)}$ such that

$$|g(s) - g(t)| < \frac{\epsilon}{2K + \alpha(b) - \alpha(a)} \quad \text{if } |s - t| < \delta \quad \longrightarrow (1)$$

Since $f \in R_\alpha[a, b]$, therefore there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2 \quad (\text{Riemann Condition}) \longrightarrow (2)$$

Let
$$M_i = \operatorname{Sup}\{f(x) : x_{i-1} \leq x \leq x_i\}$$
$$m_i = \operatorname{Inf}\{f(x) : x_{i-1} \leq x \leq x_i\}$$
$$M_i' = \operatorname{Sup}\{h(x) : x_{i-1} \leq x \leq x_i\}$$
$$m_i' = \operatorname{Inf}\{h(x) : x_{i-1} \leq x \leq x_i\}$$

Dividing the numbers $i = 1, 2, 3, \ldots, n$ into two classes A & B as under

$$A = \{i \mid 1 \leq i \leq n \text{ and } M_i - m_i < \delta\}$$

$$B = \{i \mid 1 \leq i \leq n \text{ and } M_i - m_i \geq \delta\}$$

if $i \in A$, we have by (1)

$$M_i' - m_i' < \frac{\epsilon}{2K + \alpha(b) - \alpha(a)}$$

If $i \in B$, we have from $|g(t)| < K \quad \forall t \in [a, b]$

$$\Rightarrow -K < g(t) < K$$

$$\Rightarrow 0 < g(t) \leq 2K$$

and hence $M_i' - m_i' < 2K$

Now

$$\delta \leq M_i - m_i \quad \forall i \in B$$

$$\Rightarrow \sum_{i \in B} \delta \Delta\alpha_i \leq \sum_{i \in B} (M_i - m_i) \Delta\alpha_i \leq \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i$$

$$= U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$$

$$\Rightarrow \delta \sum_{i\in B} \Delta\alpha_i < \delta^2 \Rightarrow \sum_{i\in B} \Delta\alpha_i < \delta \quad \text{by } ② \quad \rightarrow ③$$

Therefore

$$U(P,h,\alpha) - L(P,h,\alpha) = \sum_{i=1}^{n} (M_i' - m_i')\Delta\alpha_i$$

$$= \sum_{i\in A} (M_i' - m_i')\Delta\alpha_i + \sum_{i\in B} (M_i' - m_i')\Delta\alpha_i$$

$$< \frac{\epsilon}{2k+\alpha(b)-\alpha(a)} \sum_{i\in A} \Delta\alpha_i + 2k \sum_{i\in B} \Delta\alpha_i$$

$$< \frac{\epsilon}{2k+\alpha(b)-\alpha(a)} \sum_{i\in A} \Delta\alpha_i + 2k\delta$$

$$< \frac{\epsilon}{2k+\alpha(b)-\alpha(a)} \sum_{i=1}^{n} \Delta\alpha_i + \cancel{2k\delta} \quad \because \sum_{i\in A} \Delta\alpha_i < \sum_{i=1}^{n} \Delta\alpha_i$$

$$< \frac{\epsilon}{2k+\alpha(b)-\alpha(a)} [\alpha(b)-\alpha(a)] + 2k \cdot \frac{\epsilon}{2k+\alpha(b)-(\alpha)}$$

$$= \epsilon \left[\frac{2k+\alpha(b)-\alpha(a)}{2k+\alpha(b)-\alpha(a)} \right] = \epsilon$$

$\Rightarrow U(P,h,\alpha) - L(P,h,\alpha) < \epsilon$

$\Rightarrow$ Riemann's condition is true and hence $h \in R_\alpha[a,b]$

Explaination # We have taken $\delta < \frac{\epsilon}{2k+\alpha(b)-\alpha(a)}$ so that at the end we may get ϵ in the calculation. Since when a function f is uniform Continuous on $[a,b]$, for any $\epsilon > 0$, we always have a $\delta > 0$ such that

$|f(x) - f(y)| < \epsilon \rightarrow ①$ $\forall\ x, y \in [a, b]$ s.t $|x-y| < \delta_1$

Now if we choose any δ less that δ_1, then for $x, y \in [a, b]$ with $|x-y| < \delta$ Condition ① also holds because in this case we also have $|x-y| < \delta < \delta_1$.

So once a δ is chosen we can select any other δ_1 less than δ and this will also work for the same ϵ.

Corollary # If $f \in R_\alpha[a\ b]$, then $f^2 \in R_\alpha[a. b]$

Proof # Let $f \in R_\alpha[a, b]$.

$\because\ g(x) = x^2$ is continuous

$\therefore\ g \circ f \in R_\alpha[a\ b]$

$\Rightarrow g \circ f = f^2 \in R_\alpha[a\ b]$

Reduction to a Rieman Integral

The next Theorem permits us to replace the symbol $d\alpha(x)$ by $\alpha'(x)dx$ in the integral $\int_a^b f d\alpha$ when ever α' is Riemann integrable on $[a, b]$ and $f\alpha'$ is also Riemann integrable on $[a\ b]$. In this case the Riemann Stieltjes integral reduces to ordinary Riemann Integral

Theorem # Assume α increases monotonically and $\alpha' \in R$ on $[a, b]$. Let f be a bounded real function on $[a\ b]$. Then $f \in R_\alpha[a\ b]$ iff $f\alpha' \in R$ on $[a, b]$.

In this case

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$$

Proof# Since $\alpha' \in R$ on $[a\ b]$

$\therefore$ By Riemann's Condition for a given $\epsilon > 0$ there exists a partition $P = \{x_0, x_1, x_2 - - - x_n\}$ of $[a\ b]$ such that

$$U(P, \alpha') - L(P, \alpha') < \epsilon \longrightarrow ①$$

$\because$ Differentiable function is continuous and α' exists

$\therefore$ α is continuous on $[a, b]$ and hence on $[x_{i-1}, x_i]$ and its derivative also exists.

So Mean value theorem is applicable on $[x_{i-1}, x_i]$ and there exist points $t_i \in]x_{i-1}, x_i[$ such that

$$\frac{\alpha(x_i) - \alpha(x_{i-1})}{x_i - x_{i-1}} = \alpha'(t_i) \quad \forall i = 1, 2 - - - n.$$

$$\Rightarrow \alpha(x_i) - \alpha(x_{i-1}) = (x_i - x_{i-1})\alpha'(t_i) \quad \forall i$$

$$\Rightarrow \Delta\alpha_i = \alpha'(t_i)\Delta x_i \quad \forall i$$

If $s_i \in [x_{i-1}, x_i]$, then $\alpha'(t_i), \alpha'(s_i)$ both lie in $[m_i, M_i]$, where

$$M_i = \operatorname{Sup} \alpha'(x) \quad \text{on } [x_{i-1}, x_i]$$
$$m_i = \operatorname{Inf} \alpha'(x) \quad \text{on } [x_{i-1}, x_i]$$

So we have

$$|\alpha'(s_i) - \alpha'(t_i)| \leq M_i - m_i$$

$$\Rightarrow \sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i \leq \sum (M_i - m_i)\Delta x_i$$

$$\Rightarrow \sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i \leq \sum M_i \Delta x_i - \sum m_i \Delta x_i$$
$$\leq U(P, \alpha') - L(P, \alpha') < \epsilon \quad \text{By ①}$$

$$\Rightarrow \sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i < \epsilon \longrightarrow ②$$

$\because$ f is bounded on $[a\ b]$

$\therefore$ Let $|f(x)| \leq M$ where $M > 0$
$\forall x \in [a,b]$

Step-II Consider

$$\left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i \right|$$

$$= \left| \sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i - \sum f(s_i)\alpha'(s_i)\Delta x_i \right|$$

$\because \Delta\alpha_i = \alpha'(t_i)\Delta x_i$

$$= \left| \sum_{i=1}^{n} f(s_i)\left[\alpha'(t_i) - \alpha'(s_i)\right]\Delta x_i \right|$$

$$\leq M \sum_{i=1}^{n} |\alpha'(t_i) - \alpha'(s_i)|\Delta x_i$$

By property of modulus & By $f(x) \leq M$

$$< M\epsilon \qquad \text{By } ②$$

$$\Rightarrow \left| \sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum f(s_i)\alpha'(s_i)\Delta x_i \right| < M\epsilon \longrightarrow ③$$

$$\Rightarrow -M\epsilon < \sum f(s_i)\Delta\alpha_i - \sum f(s_i)\alpha'(s_i)\Delta x_i < M\epsilon \rightarrow ④$$

$$\Rightarrow \sum_{i=1}^{n} f(s_i)\Delta\alpha_i < f(s_i)\alpha'(s_i)\Delta x_i + M\epsilon$$

for all choices of s_i in $[x_{i-1}, x_i]$

So that

$$U(P,f,\alpha) < U(P, f\alpha') + M\epsilon \longrightarrow ⑤$$

Again from ④

$$-M\epsilon < \sum f(s_i)\Delta\alpha_i - \sum f(s_i)\alpha'(s_i)\Delta x_i$$

$$\Rightarrow \sum f(s_i)\alpha'(s_i)\Delta x_i < \sum f(s_i)\Delta\alpha_i + M\epsilon$$

for all choices of s_i in $[x_{i-1}, x_i]$

So that

$$U(P, f\alpha') < U(P,f,\alpha) + M\epsilon \rightarrow ⑥$$

From ⑤ and ⑥, we have

$$|U(P,f,\alpha) - U(P,f\alpha')| < M\epsilon$$

and this remains true if P is replaced by any refinement because ① remains true for any refinement of P. Therefore we have

$$\left| \overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f(x)\alpha'(x)\,dx \right| < M\epsilon$$

Since ϵ is an arbitray number and may be very small. Therefore

$$\left| \overline{\int_a^b} f\,d\alpha - \overline{\int_a^b} f(x)\alpha'(x)\,dx \right| = 0$$

$$\Rightarrow \overline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f(x)\alpha'(x)\,dx \longrightarrow ⑦$$

Similarly from ④ we can prove that

$$\underline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f(x)\alpha'(x)\,dx \longrightarrow ⑧$$

Hence if $f \in R_\alpha[a\ b]$, then $\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$

and from ⑦ and ⑧, we have

$$\overline{\int_a^b} f(x)\alpha'(x)\,dx = \underline{\int_a^b} f(x)\alpha'(x)\,dx$$

$$\Rightarrow f\alpha' \in R$$

Conversely if $f\alpha' \in R$, Then from ⑦ & ⑧, we have

$$\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$$

$$\Rightarrow f \in R_\alpha[a\ b]$$ Proved

Remarks # ∵ The product of two Riemann integrable functions is a Riemann integrable function

∴ The statement of the theorem may be as

" If f and α' are Riemann integrable on $[a,b]$, then $f\alpha' \in R[a\ b]$, and $f \in R_\alpha[a.\ b]$

other statements are

* If f is continuous and α is continuously differentiable on $[a,b]$, then $f \in R_\alpha[a\ b]$ and $f\alpha' \in R[a\ b]$

* Let $f \in R(\alpha)$ on $[a\ b]$ and α is continuously differentiable on $[a\ b]$. Then the Riemann integral $\int_a^b f\alpha' dx$ exists and we have

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\alpha'(x)\, dx$$

Change of Variable in Riemann-Stieltjes Integral

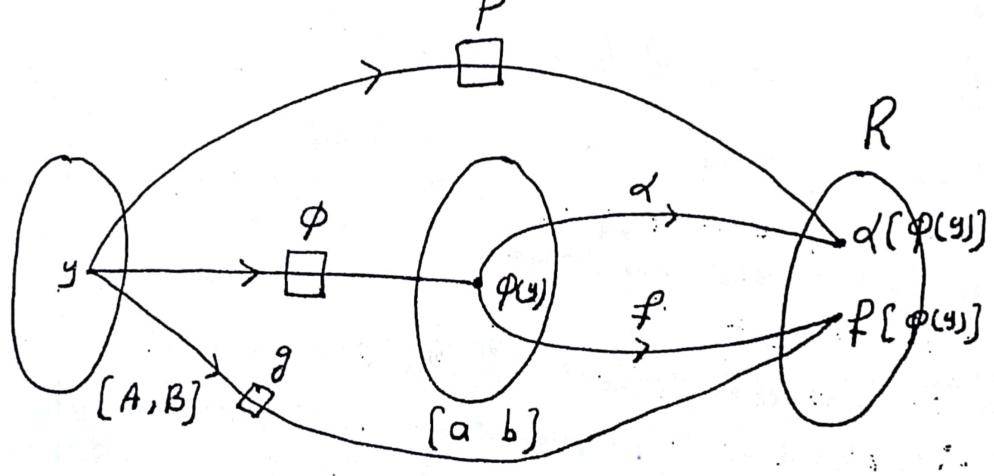
Theorem # Let $f \in R_\alpha[a,b]$ and ϕ be a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a\ b]$. Suppose α is monotonically increasing on $[a\ b]$. Let β and g be composite functions defined on $[A\ B]$ as

$$\beta(y) = \alpha(\phi(y)) \qquad g(y) = f(\phi(y)) \quad \forall y \in [A\ B]$$

Then $g \in R(\beta)$ on $[A, B]$ and

$$\int_A^B g\, d\beta = \int_a^b f\, d\alpha$$

Proof #



Note that β is a composition of α and φ and g is a composition of f and φ as shown and given

$\because$ φ is strictly increasing on $[A, B]$

$\therefore$ φ is one-to-one and has strictly increasing inverse φ^{-1} on $[a, b]$. Therefore for every partition $P = \{x_0, x_1, \dots x_n\}$ of $[a\ b]$, there corresponds one and only one partition $Q = \{y_0, y_1, \dots y_n\}$ of $[A, B]$ so that

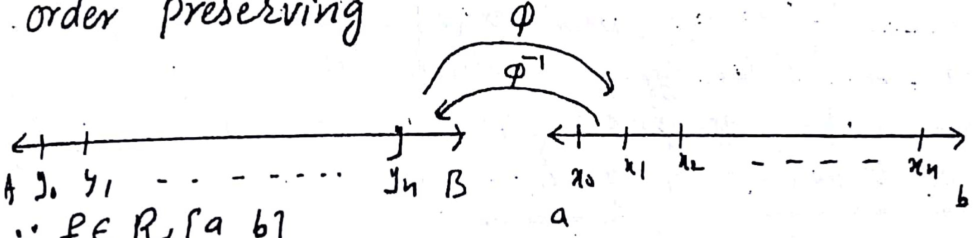
$$x_i = \varphi^{-1}(y_i)$$

Actually we can write

$$P = \phi(Q) \ \& \ Q = \phi^{-1}(P)$$

All partitions of $[A, B]$ are obtained in this way.

Also a refinement of Q produces a corresponding refinement of P and converse is also true i.e correspondence is order preserving



$\because$ $f \in R_\alpha[a\ b]$

$\therefore$ If $\epsilon > 0$, then there is a partition P_0 of $[a\ b]$ such for any refinement P of P_0, we have

$$\left| S(P, f, \alpha) - \int_a^b f d\alpha \right| < \epsilon \longrightarrow (1)$$

Let $P_0' = \varphi^{-1}(P_0)$ be the corresspmding partition of $[A\ B]$. For any refinement $P' = \{y_0, y_1, y_2 \dots y_n\}$ of P_0', the partition $P = \varphi(P') = \{x_0, x_1, \dots x_n\}$ is a refinement of P_0. Choose an arbitrary t_i in $[y_{i-1}, y_i]$. Then $s_i = \varphi(t_i)$ is a point in the interval $[x_{i-1}, x_i]$.

We have

$$S(P', g, \beta) = \sum_{i=1}^{n} g(t_i)[\beta(t_i) - \beta(t_{i-1})]$$

$$= \sum_{i=1}^{n} f[\varphi(t_i)]\{\alpha(\varphi(t_i)) - \alpha(\varphi(t_{i-1}))\}$$

$$= \sum_{i=1}^{n} f(\delta i)\{\alpha(x_i) - \alpha(x_{i-1})\}$$

$$= S(P, f\alpha) \qquad \because \delta i \in [x_{i-1}\ x_i]$$

Therefore from ①, we have

$$\left| S(P', g\ \beta) - \int_a^b f d\alpha \right| < \epsilon$$

For any refinement P' of P_o'

$\Rightarrow g \in R_\beta[A, B]$ and

$$\int_a^b f d\alpha = \int_A^B g d\beta \qquad \text{Proved.}$$

Remarks# This Theorem applies in particular to Riemann integrals i.e when $\alpha(x) = x$. Another theorem of this type in which ϕ is not required to be monotonic will later be proved for Riemann integrals:

Function Defined by Definite Integrals

Let f be Riemann integrable on $[a\ b]$, then the function F given by

$$F(x) = \int_a^x f(t)\,dt \qquad \forall x \in [a\ b]$$

is well defined because for each $x \in [a, b]$ $f \in R[a\ x]$ and as such $F(x)$ is uniquely defined on $[a\ b]$

The function F may be called the integral function of f.

We examine certain properties of this function F defined on $[a\ b]$

Theorem # (First Fundamental Theorem)
Let $f \in R[a,b]$ i.e. f is Riemann integrable on $[a\ b]$. Then function F defined by

$$F(x) = \int_a^x f(t)\,dt \quad \forall x \in [a\ b]$$

is continuous on $[a\ b]$; further, if f is continuous at a point x_0 of $[a\ b]$, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0)$$

Proof # $\because$ f is Riemann Integrable on $[a.b]$
$\therefore$ f is bounded on $[a\ b]$
Let $|f(t)| \leq M \quad \forall t \in [a\ b]$

Let us choose any two points in $[a.b]$ such that
$a \leq x < y \leq b$.

$$F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_a^y f(t)\,dt + \int_x^a f(t)\,dt$$

$$= \int_x^a f(t)\,dt + \int_a^y f(t)\,dt$$

$$= \int_x^y f(t)\,dt$$

$$\Rightarrow |F(y) - F(x)| = \left|\int_x^y f(t)\,dt\right| \leq M(y-x)$$

($\because$ If f is integrable in $[a.b]$ and $|f(x)| \leq K \ \forall x \in [a.b]$, then $\left|\int_a^b f(x)\,dx\right| \leq K|b-a|$)

For any ϵ such that

$M|y-x| < \epsilon$

where $|y-x| < \epsilon/M$

Then

$|F(y)-F(x)| < \epsilon$ whenever $|y-x| < \frac{\epsilon}{M} = \delta$

so that for any $\epsilon > 0$, we can always choose $\delta = \epsilon/M$ such that

$|F(y) - F(x)| < \epsilon$ whenever $|y-x| < \delta$

$\Rightarrow$ F is uniformly continuous on $[a, b]$.

Now a uniformly continuous function on $[a\ b]$ is continuous on $[a\ b]$

Hence F is continuous on $[a\ b]$

Differentiability at x_0

Suppose that f is continuous at x_0.

Then by definition of continuity for $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(t) - f(x_0)| < \epsilon \longrightarrow (1) \quad \text{whenever } |t-x_0| < \delta$$

$$\Rightarrow f(x_0) - \epsilon < f(t) < f(x_0) + \epsilon \quad x_0 - \delta < t < x_0 + \delta$$

$\Rightarrow$ Now

$$\frac{F(t) - F(x_0)}{t - x_0} - f(x_0)$$

$$= \frac{t}{t - x_0} \int_{x_0}^{t} f(t)\, dt - f(x_0)$$

$$= \frac{1}{t - x_0} \left[\int_{x_0}^{t} f(t)\, dt - (t - x_0) f(x_0) \right]$$

$$= \frac{1}{t-x_0}\left[\int_{x_0}^{t} f(t)\,dt - f(x_0)\int_{x_0}^{t} dt\right]$$

$$= \frac{1}{t-x_0}\int_{x_0}^{t}\{f(t) - f(x_0)\}\,dt$$

$$\Rightarrow \left|\frac{F(t)-F(x_0)}{t-x_0} - f(x_0)\right| = \frac{1}{|t-x_0|}\left|\int_{x_0}^{t}\{f(t)-f(x_0)\}dt\right|$$

$$\leq \frac{1}{|t-x_0|}\int_{x_0}^{t}|f(t)-f(x_0)|\,dt \rightarrow ②$$

($\because$ When $f_1 \leq f_2$ on $[a,b]$, then $\int_a^b f_1 dx \leq \int_a^b f_2 dx$)
and $\left|\int_a^b f dx\right| \leq \int_a^b |f| dx$

We have from ①

$$|f(t) - f(x_0)| < \epsilon \quad \text{when } |t-x_0| < \delta$$

$$\Rightarrow \int_{x_0}^{t}|f(t) - f(x_0)|\,dt < \int_{x_0}^{t}\epsilon\,dt$$

$\because$ If $f_1 \leq f_2$ on $[a,b]$ then $\int_a^b f_1 dx \leq \int_a^b f_2 dx$

using this in ②, we have

$$\left|\frac{F(t)-F(x_0)}{t-x_0} - f(x_0)\right| < \frac{1}{|t-x_0|}\int_{x_0}^{t}\epsilon\,dt$$

$$= \frac{1}{|t-x_0|}\,\epsilon\,|t-x_0| = \epsilon$$

Provided $0 < |t-x_0| < \delta$

Hence we conclude that F is differentiable at x_0 and

$$\lim_{t\to x_0}\frac{F(t)-F(x_0)}{t-x_0} = f(x_0)$$

$$\Rightarrow F'(x_0) = f(x_0)$$

OR

Differentiability of $F(x) = \int_a^x f(t)dt$ at x_0 can also be proved as under

We use the concept that if f is a continuous function on $[a\ b]$, then there exists a point c between a & b such that

$$\int_a^b f(x)dx = (b-a) f(c)$$

$$F(x_0+h) - F(x_0) = \int_{x_0}^{x_0+h} f(t)dt$$

$$= (x_0+h-x_0) f(x_0+\theta h) \qquad 0<\theta<1$$

$$= h f(x_0+\theta h)$$

Since f is continuous, we have

$$\lim_{h\to 0} f(x_0+\theta h) = f(x_0)$$

Hence $\displaystyle \lim_{h\to 0} \frac{F(x_0+h)-F(x_0)}{h} = \lim_{h\to 0} f(x_0+\theta h)$

$$\Rightarrow \lim_{h\to 0} \frac{F(x_0+h)-F(x_0)}{h} = f(x_0)$$

$$\Rightarrow \quad F'(x_0) = f(x_0)$$

$\Rightarrow$ F is differentiable at x_0 and $F'(x_0) = f(x_0)$

Primitive # If there exists function F defined on $[a, b]$ such that $F'(x) = f(x)$ on $[a.b]$, then F is called primitive or anti-derivative of f on $[a.b]$. By above theorem we note that if f is continuous on $[a.b]$, it has a primitive on $[a, b]$ and privitive is

$$\int_a^x f(t)dt$$

Results from above Theorem # From the above theorem we have following important results.

* We note that function $F(x) = \int_a^x f(t)\,dt$ is always continuous, though $f(x)$ may not be continuous.
* While f may be continuous without necessarily being differentiable at a point, the continuity of f ensures the differentiability of F. If f is differentiable at a point, then F is twice differentiable there.
* Every continuous function $f(x)$ is derivative of a continuous function given by

$$\int_a^x f(t)\,dt$$

and $$\frac{d}{dx}\left\{ \int_a^x f(t)\,dt \right\} = f(x)$$

Theorem# Suppose that F be a primitive of a continous function f on $[a.b]$. A function G defined on $[a.b]$ is also a primitive of f on $[a.b]$ iff for some constant C, $G(x) = F(x) + C$ for all x in $[a.b]$

Proof# Suppose F and G are primitive of f. Then

$$F'(x) = f(x) \quad \forall x \in [a.b]$$
$$G'(x) = f(x) \quad \text{〃} \quad \text{〃}$$

$$\Leftrightarrow \quad F'(x) = G'(x)$$

$$\Leftrightarrow \quad F'(x) - G'(x) = 0$$

$$\Leftrightarrow \quad (F' - G')(x) = 0$$

$$\Leftrightarrow \quad (F - G)'(x) = 0$$

$$\Leftrightarrow \quad F - G = C$$
$$G - F = C_1$$
$$\Leftrightarrow \quad G(x) = F(x) + C_1$$

(Proved)

Theorem # (Cauchy's Fundamental Theorem of Calculus)

If $f \in R$ on $[a,b]$ and there is a differentiable function F on $[a,b]$ such that $F' = f$, Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

OR

If f is continuous on $[a.b]$ and if F is a primitive of f on $[a\ b]$, Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$\because$ A function continuous on $[a\ b]$, is integrable because every continuous fn is integrable.

Proof #

$\because f \in R$ on $[a, b]$

$\therefore$ For a given $\epsilon > 0$. there exists a partition $P = \{x_0, x_1, x_2 - - - - x_n\}$ of $[a, b]$ such that

$$U(P.f) - L(P.f) < \epsilon \quad \to ①$$

$\because F$ is differentiable on $[a,b]$

$\therefore$ By Mean value Theorem there exists points $t_i \in\,]x_{i-1}, x_i[$ such That

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(t_i) = f(t_i) \quad \forall i = 1, 2, \dots n$$

$$\Rightarrow F(x_i) - F(x_{i-1}) = f(t_i)\,\Delta x_i \quad \forall i$$

Summing over all the intervals

$$\sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} f(t_i)\,\Delta x_i$$

$$F(b) - F(a) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i$$

(telescoping sum)

This will be true for any partition of $[a, b]$.
Now under the Condition ①. For any points t_i in $[x_{i-1}, x_i]$, we have

$$\left| \sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f\,dx \right| < \epsilon$$

$$\Rightarrow \left| F(b) - F(a) - \int_a^b f\,dx \right| < \epsilon$$

Since it holds for any arbitrary $\epsilon > 0$
Therefore

$$\int_a^b f\,dx = F(b) - F(a) \quad \text{proved.}$$

Corollary # If is differentiable on $[a, b]$ and if f' is integrable on $[a, b]$, for all x in $[a, b]$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Actually we can write

$$\int_a^x f'(x)\,dx = f(x) - f(a)$$

Remarks # Given an integrable function f on $[a \cdot b]$, it may not be possible to find an anti-derivative G of f on the entire interval. If there is none; then the theorem does not help us. On the other hand if f is continuous on $[a, b]$, then $F(x) = \int_a^x f\,dt$ is already an anti-derivative of f, but this theorem helps us evaluate the integral only if we can identify F explicitly in terms of known functions. The question whether a primitive exists and the question of the existence of an integral of function $f(x)$ in $[a, b]$ are entirely independent questions.

Change of Variable in a Riemann Integral#

The formula $\int_A^B g\, d\beta = \int_a^b f\, d\alpha$.

Previously proved for change of variable in an integral assumes the form
$$\int_a^b f(x)\,dx = \int_A^B f[g(t)]\, g'(t)\,dt$$

When $\alpha(x) = x$ and when g is a strictly monotonic function with a continuous derivative g. It is valid if $f \in R$ on $[a,b]$. When f is continuous we remove the restriction that g is monotonic

Theorem# Suppose that u has continuous derivative on $[c,d]$. Let f be continuous on the range of u. Let $u(c) = a$, $u(d) = b$. Then
$$\int_a^b f(x)\,dx = \int_c^d f(u(t))\, u'(t)\,dt$$

OR

Let u be continuously differentiable function on $I = [c,d]$. Let $u(c) = a$ $u(d) = b$. Let f be continuous on the $u(I)$ i.e range of u, then
$$\int_a^b f(x)\,dx = \int_c^d f(u(t))\, u'(t)\,dt.$$

Proof# $\because$ We know that if a function f is continuous on an interval I, then $f(I)$ is also an interval and if I is closed and bounded, then so is $f(I)$

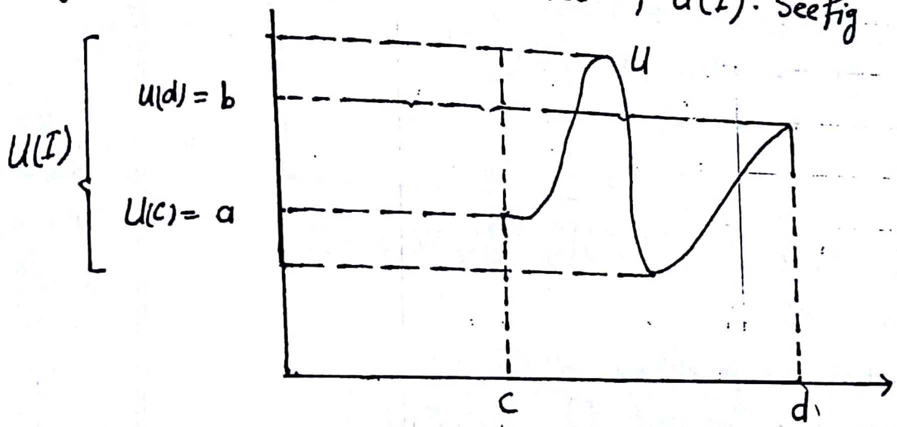
$\therefore$ u being continuous on an interval $[c,d]$ has range which is also a closed interval.

Since composite function of two continuous function is

also continuous, therefore $f(u)$ is continuous on $[c,d]$.
Again since a continuous function on an interval is integrable on that interval, therefore fou is integrable on $[c,d]$.

Also $(fou)u'$ being a product of two continuous functions is continuous on $[c,d]$.

Note that u is not assumed to be monotone so that $u(I)$ contains, but may not equal, the interval with end points $u(c)=a$ and $u(d)=b$. However since f is assumed continuous on the interval $u(I)$, it is integrable on every sub-interval of $u(I)$. See fig



For all $x \in u(I)$, define

$$F(x) = \int_a^x f(t)\,dt \longrightarrow ①$$

Also define G on $[c,d]$ as follows

$$G(x) = \int_c^x f[u(t)]\,u'(t)\,dt \longrightarrow ②$$

We first show that.

$$F(u(x)) = G(x)$$

$\because$ $f[u]u'$ is continuous

$\therefore$ By property of integral function derivative of G is equal to $f(u)u'$ on $[c,d]$ i.e

$$G'(x) = f[u(x)]\,u'(x) \quad \forall x \in [c,d] \longrightarrow ③$$

But by chain rule

$$F'[u(x)] = f[u(x)]u'(x) \quad \forall x \in [c, d] \rightarrow (4)$$

By (3) & (4), we have

$$G'(x) = F'[u(x)] \quad \forall x \in [c, d]$$

$$\because F'(x) = f(x)$$

$$\cancel{\Rightarrow G'(x) = F'(x) = f(x)}$$

~~$\Rightarrow$ G and F are two primitive of $f(x)$~~

$$\cancel{\Rightarrow G'(x) - F'(x) = 0}$$

$$\cancel{(G - F)'(x) = 0}$$

$$\cancel{\Rightarrow (G - F)(x) = \text{Constant}} \rightarrow (5)$$

~~But at $x = c$, we get from (2)~~

$$\cancel{G(c) = 0}$$

~~and~~

$\therefore$ $G(x)$ and $F[u(x)]$ are two primitives of $f(x)$ on $[c, d]$

But any two primitives differ by a constant.

So there exists a constant D such that

$$F[u(x)] = G(x) + D \quad \text{i-e} \rightarrow (5)$$

$$F(u(x)) = \int_a^{u(x)} f(t)\,dt = \int_c^{x} f[u(t)]u'(t)\,dt + D$$

Let $x = c$ so that $u(c) = a$ and we obtain

$$\int_a^a f(t)\,dt = \int_c^c f[u(t)]u'(t) + D$$

$$0 = 0 + D$$

$$0 = D$$

putting in (5), we have

$$F(u(x)) = G(x) \quad \forall x \in [c, d]$$

So $\quad F[u(d)] = G(d)$

$\Rightarrow \quad F(b) = G(d)$

$\Rightarrow \quad \int_a^b f(t)\,dt = \int_c^d f[u(t)]\,u'(t)\,dt.$

or $\quad \int_a^b f(x)\,dx = \int_c^d f[u(t)]\,u'(t)\,dt$

$\boxed{OR}$

Integration by Parts

Theorem # If F and G are primitives of f and g respectively and f, g are Riemann-integrable functions on $[a,b]$, then

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,dx$$

Proof # $\because$ Primitive of a function is continuous
$\therefore$ F, G are continuous on $[a,b]$
Again a continuous function on $[a,b]$ is Riemann-integrable on $[a,b]$ therefore F, G are Riemann-integrable on $[a,b]$.
$\because$ Product of two Riemann integrable function is Riemann integrable.
$\therefore$ Fg and fG are integrable on $[a,b]$
$\Rightarrow$ The two integral in the Theorem exist.

Let $H(x) = F(x)G(x)$

$$H'(x) = F'(x)G(x) + F(x)G'(x)$$
$$= f(x)G(x) + F(x)g(x) = h(x) \text{ say}$$

$\because$ F, G are Riemann integrable

$\therefore$ $H = FG$ is also Riemann Integrable

Also fG and Fg are Riemann integrable and $h(x) = fG + Fg$ being sum of two Riemann integrable function is Riemann Integrable on $[a, b]$

$\because$ $H'(x) = h(x)$

$\therefore$ $H(x)$ is a primitive of $h(x)$

By fundamental theorem of Calculus, we have

$$\int_a^b h(x)dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b [f(x)G(x) + F(x)g(x)]\,dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b f(x)G(x)dx + \int_a^b F(x)g(x)dx = H(b) - H(a)$$

$$\Rightarrow \int_a^b F(x)g(x)dx = H(b) - H(a) - \int_a^b f(x)G(x)dx$$

$$= F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$$

(Proved)

Problem# Suppose $f \geq 0$ i.e f is non-negative, f is Continuous on $[a, b]$ and $\int_a^b f(x) = 0$. Prove that $f(x) = 0$ for all x in $[a, b]$. Does its Converse holds? Prove your assertion.

Sol# Since f is Continuous on $[a, b]$, it is integrable. Suppose there were to exist a point x_0 in $[a\ b]$ where $f(x_0) > 0$ (i.e $f(x_0) \neq 0$).

Now if a function f is continuous at point c and $f(c) \neq 0$. Then f is locally bounded away from zero

i.e there exists a neighbourhood $N(c)$ of c and +ve constant m such that

$$|f(x)| \geq m > 0 \quad \forall x \in [a,b] \cap N(c)$$

$\because$ f is continuous at x_0

$\therefore$ f is bounded away at point x_0 i.e there exists a number $m > 0$ and a neighbourhood $N(x_0)$ of x_0 such that

$$0 < m \leq |f(x)| = f(x). \quad \because f(x) \geq 0 \quad \forall x \in [a,b] \cap N(x_0).$$

Let the interval $[a,b] \cap N(x_0)$ have end points c, d with $c < d$

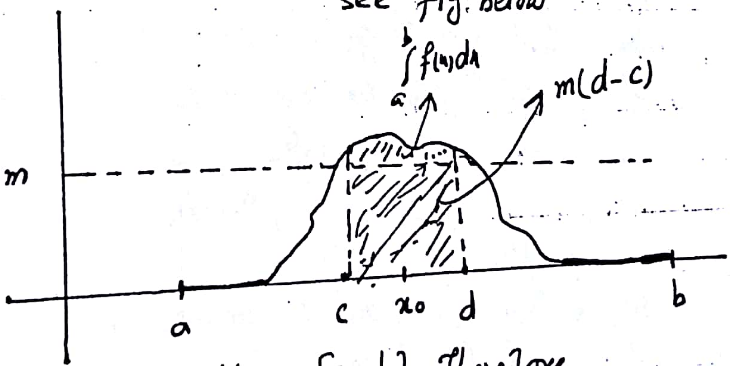
$\because$ f is continuous on $[c,d]$

$\therefore$ f is integrable on $[c,d]$

$\therefore$ $f(x) \geq m$ on $[c,d]$

$$\therefore \int_c^d f(x)dx \geq m(d-c) > 0$$

See fig. below



But $f(x) \geq 0 \quad \forall x \in [a,b]$. Therefore

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^b f(x)dx \geq \int_c^d f(x)dx > 0$$

which contradicts the hypothesis that

$$\int_a^b f(x)dx = 0$$

Therefore there can exist no point in $[a,b]$ where

where f is not zero. We conclude that f is identically zero on $[a, b]$

Its converse is also true i.e if $f(x) \geq 0$ and $f(x) = 0$ $\forall x \in [a, b]$, then

$$\int_a^b f(x)dx = 0$$

$\because$ $f(x) \geq 0$ and $f(x) = 0$

$\therefore$ $U(P, f) = 0 = L(P, f)$ $\forall P$

Therefor

$$\int_a^b f(x)dx = 0$$

Another Question

Here raises another question. Suppose that f and g are two bounded functions each of which is integrable on $[a, b]$ and suppose further that $f(x) = g(x)$ $\forall x \in [a, b]$ except a finite set of points $\{c_1, c_2 \dots c_k\}$. Can we conclude that

$$\int_a^b f(x)dx = \int_a^b g(x)dx$$

To resolve this question, consider the following simple, self-evident observation.

Lemma # If f is bounded function on $[a, b]$ such that $f(x) = 0$ $\forall x \in [a, b]$ and $f(a) \neq 0$, then f is integrable on $[a, b]$ and $\int_a^b f(x)dx = 0$

Proof # Without loss of generality, $f(a) > 0$. Let $\epsilon > 0$

Choose a +ve $\delta < \epsilon / f(a)$ and choose any partition P_0 of $[a, b]$ such that $\|P_0\| < \delta$.
Let $P = \{x_0, x_1, \ldots, x_n\}$ be any refinement.

Then $$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$= \sum_{i=1}^{n} f(a) \Delta x_i$$

$$= f(a) \sum_{i=1}^{n} \Delta x_i = f(a)\Delta x_1 + \sum_{i=2}^{n} M_i \Delta x_i$$

$$\leq f(a) \Delta x_1 + 0 \because \Sigma$$

$$= f(a) \Delta x_1 < f(a) \delta < f(a) \frac{\epsilon}{f(a)} = \epsilon$$

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$$

$$= 0 \qquad \because \text{each } m_i = 0$$

$\Rightarrow U(P, f) - L(P, f) = f(a) \Delta x_1 < \epsilon$
$\Rightarrow$ Riemann Condition is true.
$\Rightarrow$ f is Riemann Integrable function.

Further $$\overline{\int_a^b} f\,dx = \text{Inf}\{U(P, f) : P \in \mathcal{P}[a, b]\}$$
$$= 0$$

and $$\underline{\int_a^b} f\,dx = \text{Sup}\{L(P, f) : P \in \mathcal{P}[a, b]\}$$
$$= 0$$

$$\Rightarrow \int_a^b f\,dx = 0$$

Similarly if $f(x) = 0 \quad \forall x \in [a, b)$ and if $f(b) \neq 0$, then again f is $R[a, b]$ and $\int_a^b f(x)\,dx = 0$

Conclusion # We note that $f(x)=0$ at all points of $[a,b]$ except at c, i.e except at a finite number of points of $[a,b]$ i.e we have changed value of f at a finite no of points and we see that value of integral $\int_a^b f(x) = 0$. Thus we conclude that if values of function are changed at finite no of points, then values of integral is not changed.

Lemma # Suppose that f and g are two bounded, integrable functions on $[a,b]$ such that $f(x) = g(x)$ for all x in the open interval (a,b). Then

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

Proof # Let c be any point in (a,b). We have

$$\left|\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right| \le \int_a^b |f(x)-g(x)|\,dx$$

$$= \int_a^c |f(x)-g(x)|\,dx + \int_c^d |f(x)-g(x)|\,dx$$

$$= 0 + 0 = 0 \quad \because f(x)-g(x) = 0 \ \forall x \in (a,b)$$

Here $f(x)-g(x)$ is not zero at a, b only i.e at finite no of points. Hence

$$\int_a^b |f(x)-g(x)|\,dx = 0 = \int_c^d |f(x)-g(x)|\,dx$$

$$\Rightarrow \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = 0$$

$$\Rightarrow \int_a^b f\,dx = \int_a^b g(x)\,dx$$

Theorem # Suppose that f and g are two bounded functions on $[a,b]$ each of which is integrable on $[a,b]$. If $f(x) = g(x)$ except at finitely many points $c_1, c_2, \ldots c_k$, then

$$\int_a^b f(x)dx = \int_a^b g(x)dx$$

Proof # Index c_j so that

$$a = c_0 < c_1 < \cdots\cdots < c_k < c_{k+1} = b$$

Then we have

$$f(x) = g(x) \quad \forall x \in (c_{j-1}, c_j) \quad j = 1, 2, \ldots k+1$$

$\Rightarrow$ f, g are not equal at finite points i.e.

$$f(x) - g(x) = 0 \quad \forall x \in (c_{j-1}, c_j)$$

except at points $c_j, j = 1, 2, \ldots k$

Therefore by above Lemma, we have

$$\int_{c_{j-1}}^{c_j} f(x)dx = \int_{c_{j-1}}^{c_j} g(x)dx \quad \forall j = 1, 2, \ldots k+1$$

Now

$$\int_a^b f(x)dx = \sum_{j=1}^{k+1} \int_{c_{j-1}}^{c_j} f(x)dx = \sum_{j=1}^{k+1} \int_{c_{j-1}}^{c_j} g(x)dx$$

$$= \int_a^b g(x)dx$$

$$\Rightarrow \int_a^b f(x)dx = \int_a^b g(x)dx$$

Remarks # In the above Theorem g has different values at finite number of points than the values of f at those points. So we can say that g is obtained from f by changing the value of f at

at a finite number of points of $[a, b]$. Thus we restate the above theorem in another way as under

Theorem # If a function $f(x)$ is integrable on $[a, b]$, then every other function $g(x)$ obtained by altering the value of $f(x)$ at a finite number of points of $[a, b]$ is integrable on $[a, b]$ and

$$\int_a^b f(x)dx = \int_a^b g(x)dx$$

Proof # $\because$ g is obtained by f and f is bounded
$\therefore$ g is also bounded.

Let $M = \text{Sup } f(x)$ on $[a, b]$
$m = \text{Inf } f(x)$ on $[a\ b]$
$M' = \text{Sup } g(x)$ on $[a . b]$
$m' = \text{Inf } g(x)$ on $[a\ b]$

Let $c_1, c_2 \dots c_k$ be the points at which $f(x) \neq g(x)$
We can enclose these points in a finite number of intervals of total length less than a given $\epsilon > 0$.
Let $P = \{x_0, x_1, x_2 \dots x_k, x_{k+1}, \dots x_n\}$ be a partition such 1st k component intervals $[x_{i-1}, x_i]$ $i = 1, 2 \dots k$ enclose $c_i : i = 1, 2 \dots k$.
and the total length of these intervals is less than ϵ.

Let
$$M_i' = \text{Sup } g(x) \text{ on } [x_{i-1}, x_i]$$
$$m_i' = \text{Inf } g(x) \text{ on } [x_{i-1}, x_i]$$

Then $m' \leq m_i' \leq M'$
and $m' \leq M_i' \leq M'$
$\Rightarrow M_i' - m_i' \leq M' - m'$

Now

$$U(P, g) - L(P, g) = \sum_{i=1}^{n} (M_i' - m_i')\Delta x_i$$
$$= \sum_{i=1}^{k} (M_i' - m_i')\Delta x_i + \sum_{k+1}^{n} (M_i' - m_i')\Delta x_i$$

$$\leq (M'-m')\sum_{i=1}^{k}\Delta x_i + (M'-m')\sum_{i=k+1}^{n}\Delta x_i$$

$$< (M'-m')\epsilon + (M'-m')(b-a) \qquad \because \sum_{i=k+1}^{n}\Delta x_i < (b-a)$$

$$= (M'-m')[\epsilon + (b-a)] = \epsilon'$$

$\because$ ϵ is an arbitrary

$\therefore$ ϵ' is an arbitrary

Thus for any arbitrary $\epsilon' > 0$, we have found a partition P such that

$$U(P,g) - L(P,g) < \epsilon'$$

$\Rightarrow$ Riemann's Condition is true for g

$\Rightarrow$ g is Riemann Integrable on $[a,b]$

Now equality of integrals of f & g on $[a,b]$ proved as in above theorem.

Theorem# A bounded function whose discontinuities can be enclosed in a finite number of intervals whose length is less than ϵ is integrable on $[a\ b]$.

Proof# Let $P = \{x_0, x_1, x_2 - - - - - x_n\}$ such that its k component intervals enclose the discontinuities of f such that their total length $\sum_{i=1}^{k}\Delta x_i$ is less than given $\epsilon > 0$. In such intervals $M_i - m_i$ may large, but since f is bounded, therefore

$$M_i - m_i \leq M - m \quad \text{where } M, m \text{ are}$$

bounds of f on $[a,b]$.

In the sub-intervals which do not contain the discontinuities of f, this function is continuous.

$$U(P,f) - L(P,f) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i$$

$$\leq (M-m)\sum_{i=1}^{n} \Delta x_i$$

$$= (M-m)\left[\sum_{i=1}^{k} \Delta x_i + \sum_{i=k+1}^{n} \Delta x_i\right]$$

$$< (M-m)\left[\epsilon + (b-a)\right] = \epsilon' \quad \because \sum_{i=k+1}^{n} \Delta x_i < b-a$$

$\Rightarrow$ Riemann's Condition is true.
Hence f is Riemann integrable function on $[a.b]$

Example# Consider the function

$$f(x) = 0 \quad \text{when } x \text{ is an integer}$$
$$= 1 \quad \text{otherwise.}$$

in the interval $(0, m)$, where m is a +ve integer. This function is integrable because its discontinuities are finite in number being situated at points $x = 1, 2, 3, \ldots m$. If each of these points be enclosed in an interval of length less than ϵ/m. Then total length of such m-intervals is less than $m \cdot \epsilon/m = \epsilon$ and so f is integrable

Mean Value Theorems of Integral#

Theorem#1: If f is continuous on $[a.b]$ and α is increasing on $[a.b]$, there exists a point c in $[a, b]$ such that

$$\int_a^b f(x)\,d\alpha = f(c)\int_a^b d\alpha$$

$$= f(c)\left[\alpha(b) - \alpha(a)\right].$$

Proof# If $\alpha(a) = \alpha(b)$, then theorem holds trivially, both sides being zero any value of c in $[a\ b]$ gives the desired result. Hence we can assume that

$\alpha(a) < \alpha(b)$

$\because$ a Continuous function f on [a, b] attains its max. and min on [a b],

$\therefore$ f attains its sup and inf on [a, b].

Let $M =$ Sup $f(x)$ on [a, b]

$m =$ Inf $f(x)$ on [a b]

Since all upper and lower sums satisfy the relation

$$m[\alpha(b)-\alpha(a)] \leq L(P,f,\alpha) \leq U(P,f,\alpha) \leq M[\alpha(b)-\alpha(a)]$$

$$\Rightarrow m[\alpha(b)-\alpha(a) \leq L(P,f\alpha) \leq \int_a^b f d\alpha \leq U(P,f,\alpha) \leq M[\alpha(b)-\alpha(a)]$$

($\because$ f is continuous on [a b] $\therefore$ $f \in R_\alpha[a\ b]$)

$$\Rightarrow m[\alpha(b)-\alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(b)-\alpha(a)].$$

$$\Rightarrow m \leq \frac{\int_a^b f d\alpha}{[\alpha(b)-\alpha(a)]} \leq M \longrightarrow (1)$$

According to Intermediate value Theorem if is continuous on [a, b] and λ is any number between $f(a)$, $f(b)$ i.e $f(a) < \lambda < f(b)$, then there is a point x_0 (a b) such that

$$f(x_0) = \lambda$$

From (1) if $\dfrac{\int_a^b f d\alpha}{\alpha(b)-\alpha(a)}$ is equal to either of M, m, then Theorem is proved because M, m are values of f at some point of [a, b] and that point will be point c as required by theorem. If $\int_a^b f d\alpha / \alpha(b)-\alpha(a)$ not equal to m, M, Then

$$m < \frac{\int_a^b f d\alpha}{\alpha(b)-\alpha(a)} < M \rightarrow (2)$$

Now there are points $c_1, d_1 \in [a\ b]$ such that
$f(c_1) = m$ and $f(d_1) = M$

and $[c_1, d_1] \subseteq [a, b]$ if $c_1 < d_1$
, $[d_1, c_1] \subseteq [a, b]$ if $d_1 < c_1$

Then by intermediate value theorem there exists a point $c \in [c_1, d_1]$ or $c \in [d_1, c_1]$ such that

$$f(c) = \frac{\int_a^b f\,d\alpha}{\alpha(b) - \alpha(a)}$$

But then $c \in [a, b]$ and we have

$$\int_a^b f\,d\alpha = f(c)\,[\alpha(b) - \alpha(a)]$$

Theorem # If f is continuous on $[a, b]$ and if. if g is non-negative i.e $g(x) \geq 0$, $g \in R_\alpha[a.b]$, α is increasing on $[a, b]$, then there is a number $c \in [a, b]$ such that

$$\int_a^b fg\,d\alpha = f(c)\int_a^b g\,d\alpha$$

Proof # $\because$ f is continuous $[a.b]$
$\therefore$ $f \in R_\alpha[a, b]$

Also since $g \in R_\alpha[a\ b]$ and product of two Riemann stieltjes integrable function is integrable, therefore $fg \in R_\alpha[a, b]$

Again since f is continuous on $[a.b]$, therefore it attains its supremum and infimum on $[a, b]$.

Let $M = \sup f(x)$ on $[a, b]$
$m = \inf f(x)$ on $[a, b]$

Then

$$m \leq f(x) \leq M \quad \forall x \in [a,b]$$

$$\because \quad g(x) \geq 0 \quad \forall x \in [a,b]$$

$$\Rightarrow m\,g(x) \leq f(x)\,g(x) \leq M g(x) \quad \forall x \in [a,b]$$

Integrating

$$m \int_a^b g(x)\,d\alpha \leq \int_a^b f(x) g(x)\,d\alpha \leq M \int_a^b g(x)\,d\alpha$$

$$\Rightarrow m \leq \frac{\int_a^b f g\,d\alpha}{\int_a^b g(x)\,d\alpha} \leq M$$

By intermediate value Theorem there is a point $c \in [a\ b]$ such that

$$\frac{\int_a^b f g\,d\alpha}{\int_a^b g(x)\,d\alpha} = f(c)$$

$$\Rightarrow \int_a^b f g\,d\alpha = f(c) \int_a^b g(x)\,d\alpha$$

Note that the theorem also holds if $g(x) \leq 0$ for all $x \in [a,b]$

Theorem # (For Riemann Integral)

If f is continuous on $[a,b]$, then there exists a point c in $[a,b]$ such that

$$\frac{1}{b-a} \int_a^b f(x)\,dx = f(c)$$

Remarks # The quantity $\frac{1}{b-a} \int_a^b f(x)\,dx$ is called

the average value of f, on $[a,b]$, the value of f at c, is the average value of f over the interval $[a,b]$

Proof# Simply put $g(x)=1 \quad \forall x \in [a,b]$ in the above Theorem.

Theorem# Assume that α is increasing and let $f \in R(\alpha)$ on $[a,b]$. let $f \in R(\alpha)$ on $[a\ b]$ and let M, m denote respectively, the sup and inf of the set $\{f(x) \mid x \in [a,b]\}$. Then there exists a real number c satisfying $m \le c \le M$ such that

$$\int_a^b f(x)\,d\alpha(x) = c\int_a^b d\alpha(x) = c[\alpha(b)-\alpha(a)]$$

Proof# If $\alpha(a)=\alpha(b)$, the Theorem holds trivially, both sides being 0. Hence we can assume that $\alpha(a) < \alpha(b)$. Since all upper and lower sums satisfy

$$m[\alpha(b)-\alpha(a)] \le L(P,f,\alpha) \le U(P,f,\alpha) \le M[\alpha(b)-\alpha(a)]$$

$$\Rightarrow m[\alpha(b)-\alpha(a)] \le \int_a^b f\,d\alpha \le M[\alpha(b)-\alpha(a)]$$

$$\Rightarrow m \le \frac{\int_a^b f\,d\alpha}{\alpha(b)-\alpha(a)} \le M$$

Therefore quotient $c = \dfrac{\int_a^b f\,d\alpha}{\alpha(b)-\alpha(a)}$ lies between m and M. Thus $\int_a^b f\,d\alpha = c[\alpha(b)-\alpha(a)]$. When f is continuous on $[a,b]$, the intermediate value Theorem yields $c = f(x_0)$ for some $x_0 \in [a,b]$

Remarks# Mean value Theorems lik the global behavior– the value of an integral of a function over an interval–with the value of a function at a single point. Thus Mean value theorem form a bridge between global and pointwise behaviour. Also there are relatively few cases in which the explicit value of the integral can be obtained. However it is often sufficient to have an estimate for integral rather than its exact value.

Theorem# (Second Mean Value Theorem for Riemann-Stieltjes integrals)

Assume that α is continuous and that f is monotonically increasing on $[a,b]$. Then there exists a point x_0 in $[a,b]$ such that

$$\int_a^b f(x)\,d\alpha(x) = f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^b d\alpha(x)$$

Proof# $\because$ α is continuous on $[a,b]$.

$\therefore$ α is $R_f[a,b]$

Also $f \in R_\alpha[a,b]$ and by Theorem of by parts integration we have

$$\int_a^b f(x)\,d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x)\,df(x) \quad \rightarrow (1)$$

Now in integral $\int_a^b \alpha(x)\,df$ α is continuous and f is monotonically increasing, therefore by above mean value theorem there exists a point $x_0 \in [a,b]$ such that

$$\int_a^b \alpha(x)\,df(x) = \alpha(x_0)[f(b)-f(a)]$$

using it in (1)

$$\int_a^b f\,d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - [\alpha(x_0)\{f(b) - f(a)\}]$$

$$= f(b)\alpha(b) - f(a)\alpha(a) - \alpha(x_0)f(b) + \alpha(x_0)f(a)$$

$$= f(a)[\alpha(x_0) - \alpha(a)] + f(b)[\alpha(b) - \alpha(x_0)]$$

$$= f(a)\int_a^{x_0} d\alpha(x) + f(b)\int_{x_0}^{b} d\alpha(x)$$

Corollary # If f is continuous and monotone increasing on $[a,b]$, there exists a point c in $[a,b]$ such that

$$\int_a^b f(t)\,dt = f(a)(c-a) + f(b)(b-c)$$

Proof # In the above theorem take ~~$\alpha(t)$~~ $f(t)$ and $\alpha(t) = t$ ~~$g(t) = t$~~ in place of α, then for $x_0 = c$.

$$\int_a^b \cancel{f(t)\,dg(t)} =$$

$$\int_a^b f(t)\,d\alpha(t) = f(a)(c-a) + f(b)(b-c)$$

$$\int_a^b f(t)\,dt = f(a)(c-a) + f(b)(b-c)$$

Functions of Bounded Variation

Introduction # We discuss a class of functions on $[a,b]$ which are called function of bounded variation on $[a,b]$. This class of functions is closely related to monotonic functions. These functions are intimately connected with curves having finite length (Rectifiable curves). The functions of bounded variation play an important role in various parts of analysis such as Riemann-Stieltjes integration, Rectifiable curves and fourier series.

Intuitively, a function f is of bounded variation if its total amount of wiggling remains bounded.

Function of Bounded Variation # & Total Variation of a Function

Let f be a function defined on $[a,b]$ and $P=\{a=x_0, x_1, x_2, \dots, x_n=b\}$ be a partition of $[a,b]$. f is said to be of bounded variation if the set of sums of the form

$$\sum_{i=1}^{n} |f(x_i)-f(x_{i-1})| = \sum_{i=1}^{n} |\Delta f_i|$$

is bounded. i.e there exists a number $M>0$ such that

$$\sum_{i=1}^{n} |\Delta f_i| \leq M \quad \text{for all partitions } P \text{ of } [a,b].$$

i.e if the set of sums is bounded.

The set of all functions of bounded variation on $[a,b]$ is denoted by $BV[a,b]$

Total Variation of Function

If $f \in BV(a,b)$, then the set of real numbers

$$V = \left\{ \sum_{i=1}^{n} |\Delta f_i| : \forall P \in \mathcal{P}[a,b] \right\}$$

is bounded above and according to Completeness property of the set of real numbers, V attains its sup in R. The supremum of this set is called total variation of f on $[a,b]$ and is denoted by $V(f;a,b)$ or $V(f,[a,b])$ or $V_f(a,b)$ or $\overset{b}{\underset{a}{V}} f$ or $V(f,a,b)$. Thus

$$V(f;a,b) = \operatorname{Sup}\left\{ \sum_{i=1}^{n} |\Delta f_i| : P \in \mathcal{P}[a,b] \right\}$$

Remarks # * $f(x_i) - f(x_{i-1}) = \Delta f_i$ denotes the change or variation of f, on the ith partition interval $[x_{i-1}, x_i]$ and $|\Delta f_i|$ is numerical value of this change.

* $\sum_{i=1}^{n} |\Delta f_i|$ represents the sum of variations of f, corresponding to all partition intervals of a partition in numerical sense.

* The function f is said to be of bounded variation on $[a,b]$ iff its total variation is finite i.e $V(f;a,b) < \infty$.

* If there is no danger of misunderstanding, we write V_f instead of $V_f(a,b)$ and $V(f)$ instead of $V(f,a,b)$

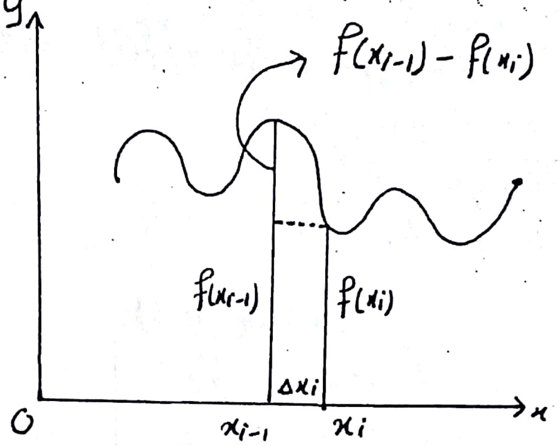
* Total variation $V(f;a,b)$ depends upon

f, a, and b

* $V_f(a,b) \geq 0$ because $\sum_{i=1}^{n} |\Delta f_i| \geq 0$ V_f
* $V_f(a,b) = 0$ iff f is constant

Explaination# Modulus is involved in evaluating the sum $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$

because function f, may be increasing or decreasing in the ith interval



Note# Evidently,

$$V(f; a,b) = \text{Sup}\left\{\sum_{i=1}^{n} |\Delta f_i| : P \in \mathcal{P}[a,b]\right\}$$

$$\geq |f(b) - f(a)|$$

Theorem# A function of bounded variation is always bounded

OR

If $f \in BV(a,b)$, then f is bounded on $[a,b]$

Proof# Let f be of bounded variation on $[a,b]$ i.e let $f \in BV(a,b)$. Then there is an $M > 0$ such that

$$\sum_{i=1}^{n} |\Delta f_i| \leq M \quad \text{for all partition of } [a,b]$$

In particular choose x in (a,b) and consider

a partition $P = \{a, x, b\}$. Then

$$\sum_{i=1}^{2} |\Delta f_i| \leq M$$

$$\Rightarrow |f(x) - f(a)| + |f(b) - f(x)| \leq M \rightarrow ①$$

Now

$$|f(x)| - |f(a)| \leq |f(x) - f(a)|$$

$$\leq |f(x) - f(a)| + |f(b) - f(x)| \leq M$$

$$\Rightarrow |f(x)| - |f(a)| \leq M$$

$$\Rightarrow |f(x)| \leq M + |f(a)|$$

$\because$ x is any arbitrary point in (a, b)

$\therefore$ $|f(x)| \leq M + |f(a)| \quad \forall x \in (a, b)$

in fact

$$|f(x)| \leq M + |f(a)| \quad \forall x \in [a, b]$$

$\Rightarrow f$ is bounded on $[a, b]$.

Note #(1) For any two real numbers x & y, we have

(i) $|x - y| \geq |x| - |y|$

(ii) $|x + y| \leq |x| + |y|$

(iii) $|x - y| \leq |x| + |y|$

(iv) $|x + y| \geq |x| - |y|$

(2) If f is bounded real valued function defined on any non-empty subset S of R^n, then uniform norm of f is denoted and defined as

$$\|f\|_{\infty} = \sup\{f(x) : x \in S\}$$

Now f will be bounded if $\|f\|_{\infty}$ is finite. and in above Theorem

$$|f(x)| \leq M + |f(a)| \quad \forall x \in [a, b]$$

$\Rightarrow \|f\|_{\infty} \leq M + |f(a)|$ and f is bounded.

Theorem# If f is monotone on $[a,b]$, then f is of bounded variation on $[a,b]$ & $V_a^b f = f(b)-f(a)$ for monotone increasing function.

Proof# Let f be monotone increasing on $[a,b]$.
Then for any partition $P=\{a=x_0, x_1, x_1, \dots x_n=b\}$
$\Delta f_i \geq 0 \quad \forall i$ and

$$\sum_{i=1}^{n}|\Delta f_i| = \sum_{i=1}^{n}\Delta f_i = \sum_{i=1}^{n}[f(x_i)-f(x_{i-1})]$$

$$= f(b)-f(a) = M$$

$$\Rightarrow \sum_{i=1}^{n}|\Delta f_i| = f(b)-f(a) \quad \text{for all partitions of } [a,b]$$

$$\Rightarrow \left\{\sum_{i=1}^{n}|\Delta f_i| : P \in \mathcal{P}[a,b]\right\}$$

is bounded and hence f is of bounded variation on and

$$V(f;a,b) = f(b)-f(a)$$

If f is monotonically decreasing, then for any partition P, we have

$$\Delta f_i \leq 0 \quad \forall i$$

$$\Rightarrow f(x_i)-f(x_{i-1}) \leq 0 \quad \forall i$$

$$\Rightarrow -[f(x_{i-1})-f(x_i)] \leq 0 \quad \forall i$$

$$\Rightarrow f(x_{i-1})-f(x_i) \geq 0 \quad \forall i$$

Now $$\sum_{i=1}^{n}|\Delta f_i| = \sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$$

$$= \sum_{i=1}^{n}|f(x_{i-1})-f(x_i)|$$

$$= \sum_{i=1}^{n}[f(x_{i-1})-f(x_i)] \quad \because f(x_{i-1})-f(x_i) \geq 0$$

$= f(a) - f(b) = M$

$\Rightarrow f$ is is of bounded variation on $[a, b]$

and $V(f; a, b) = f(a) - f(b)$

Theorem# If f is continuous on $[a, b]$ and if f' exists and is bounded in (a, b), then f is of bounded variation.

Proof# Since $f'(x)$ is bounded in (a, b)
$\therefore \exists$ an $M > 0$ such that

$$|f'(x)| \leq M \qquad \forall x \in (a, b)$$

Let P be any partition of $[a, b]$. Then f is continuous on each $[x_{i-1}, x_i]$ and derivable in each (x_{i-1}, x_i). So MVT is applicable and we have for each $[x_{i-1}, x_i]$

$$\Delta f_i = f(x_i) - f(x_{i-1})$$

$$= f'(t_i)(x_i - x_{i-1}) \qquad \text{for some } t_i \in (x_{i-1}, x_i) \ \forall i$$

Thus

$$\sum_{i=1}^{n} |\Delta f_i| = \sum_{i=1}^{n} |f'(t_i)|\,|x_i - x_{i-1}|$$

$$\leq M \sum_{i=1}^{n} |x_i - x_{i-1}|$$

$$= M(b - a) \qquad \forall i$$

$\Rightarrow f$ is of bounded variation on $[a, b]$

Examples# (1) Let $f(x) = \tan^{-1} x$ on any interval $[a, b]$. Then f is continuous on $[a, b]$ and $f'(x) = \frac{1}{1+x^2}$

and $f'(x) = \frac{1}{1+x^2} \leq 1$

Hence we deduce by above Theorem that f is of bounded variation on $[a,b]$

2)) # If we consider trignometric functions $\sin x$ and $\cos x$, then we have

if $f(x) = \sin x$ on $[a,b]$

then $f'(x) = \cos x$

and $|f'(x)| \leq 1$ on $(a\ b)$

Therefore $f(x) = \sin x$ is of bounded variation on $[a\ b]$.

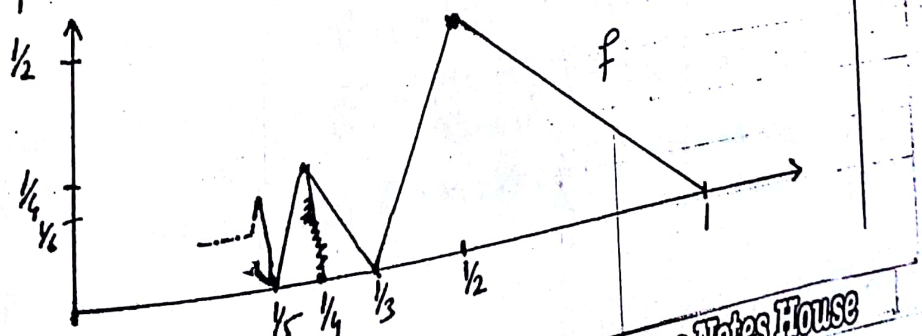
Example # The function $f(x) = \cos x$ is monotone decreasing on $[0\ \pi]$ and therefore is of bounded variation on $[0\ \pi]$

Remarks # A continuous function is not necessarily of bounded variation. We explain this by following examples.

Example 1) # Let

$$f(x) \begin{cases} \frac{1}{n} & \text{if } n \text{ is even +ve integer and } x = \frac{1}{n} \\ 0 & \text{if } n \text{ is an odd +ve integer } \& \ x = \frac{1}{n} \\ 0 & \text{if } x = 0 \text{ or } x = 1 \end{cases}$$

Let f be linear in between as in fig below.



Then f is continuous on $[0\ 1]$ but $f \notin BV[0,1]$.
For the partition

$$\left\{0, \frac{1}{n}, \frac{1}{n-1}, \dots \frac{1}{2}, \frac{1}{1}\right\}$$

where n is an odd +ve integer, we have

$$\sum_{i=1}^{n} |\Delta f_i| = |f(0) - f(\tfrac{1}{n})| + | + |f(\tfrac{1}{n}) - \tfrac{1}{n-1}| + \dots$$
$$\dots + |f(\tfrac{1}{3}) - f(\tfrac{1}{2})| + |f(\tfrac{1}{2}) - f(1)|$$

$$= 0 + \frac{1}{n-1} + \frac{1}{n-1} + \dots + \frac{1}{4} + \frac{1}{4} + \frac{1}{2} + \frac{1}{2}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{2}{n-1}$$

which is unbounded because the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

<u>Note</u> note that $\sum \frac{1}{n^p}$ is dgt if $p \le 1$ and Cgt if $p > 1$

<u>Example</u> 2 # Function $f(x)$ defined by

$$f(x) = x \sin(\pi/x) \qquad x \neq 0$$
$$= 0 \qquad x = 0$$

on $[0, 1]$

Clearly f is continuous on $[0, 1]$
Consider the partition

$$\cancel{P = \left\{0, \frac{1}{n}, \frac{1}{n-1}, \dots \frac{1}{3}, \frac{1}{2}, \frac{1}{1} = 1\right\}}$$ cancelled.

We have for partition

$$P = \left\{0, \frac{2}{2n+1}, \dots \frac{2}{7}, \frac{2}{5}, \frac{2}{3}, 1\right\}$$

$$\sum_{i=1}^{n} |\Delta f_i| = |f(1) - f(2/3)| + |f(2/3) - f(2/5)| + \dots + |f(\tfrac{2}{2n+1}) - f(0)|$$

$$= |0 + 2/3| + |2/3 + 2/5| + (2/5 + 2/7) + \dots + \frac{2}{2n+1}$$

$$= 4\left[\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2n+1}\right]$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{2n+1} = \frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots$ is divergent, therefore the sequence of its partial sums $\{S_n\}$ is not bounded i.e $\{\sum_{i=1}^{n}|\Delta f_i|\}$ is not bounded.

Hence f is not of bounded variation on $[0,1]$.

Example 3)# Consider a function

$$f(x) = x\cos\left(\frac{\pi}{2x}\right) \qquad x \neq 0$$
$$= 0 \qquad x = 0$$

on $[0,1]$

Clearly f is Continuous on $[0,1]$. But for partition

$$P = \left\{0, \frac{1}{2n}, \frac{1}{2n-1}, \cdots, \frac{1}{3}, \frac{1}{2}, 1\right\}$$

We have

$$\sum_{i=1}^{n}|\Delta f_i| = |f(1)-f(\tfrac{1}{2})| + |f(\tfrac{1}{2})-f(\tfrac{1}{3})|$$
$$+ \cdots + |f(\tfrac{1}{2n-1})-f(\tfrac{1}{2n})| + |f(\tfrac{1}{2n})-f(0)|$$

$$= \frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{6}+\frac{1}{6}+\cdots+\frac{1}{2n}+\frac{1}{2n}$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

Thus can not be bounded for all n because the series $\sum(\frac{1}{n})$ is divergent and the sequence $\{\sum_{i=1}^{n}|\Delta f_i|\}_{n=1}^{\infty}$ of partial sums of $\sum\frac{1}{n}$ is not bounded.

Hence f is not of bounded variation on $[0,1]$

Example # Consider function

$$f(x) = x\cos(\pi/x) \qquad \text{when } x \neq 0$$
$$= 0 \qquad \text{when } x = 0$$

on $[0,1]$

Clearly f is continuous on $[0,1]$

For a partition
$P = \{0, \frac{1}{n}, \frac{1}{n-1}, \ldots \frac{1}{3}, \frac{1}{2}, 1\}$

we have

$$\sum_{i=1}^{n} |\Delta f_i| = |f(1) - f(1/2)| + |f(1/2) - f(1/3)| + \cdots + |f(1/n) - f(0)|$$

$$= (1 + 1/2) + (\tfrac{1}{2} + \tfrac{1}{3}) + \cdots + (\tfrac{1}{n-1} + \tfrac{1}{n}) + (\tfrac{1}{n} + 0)$$

$$= 1 + \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \cdots \frac{2}{n}$$

$$= 1 + 2\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right)$$

which is not bounded because the series $\sum \frac{1}{n}$ is dgt.

$\Rightarrow$ f is not of bounded Variation on $[0,1]$

A function of bounded Variation may not be Continuous

It may be noted that a function of bound variation may not be continuous. For example the greatest integer function

$$f(x) = [x]$$

is of bounded variation in $[0\ 2]$ but it is not Continuous

Example # The function $f(x) = x^2 \cos(1/x)$ $x \neq 0$
$= 0$ $x = 0$

is of bounded variation on $[0,1]$, since f' is bounded on $[0,1]$. In fact $f'(0) = 0$ and

for $x \neq 0$

$$f'(x) = \sin(1/x) + 2x\cos(1/x)$$

so that

$$|f'(x)| \leq 3 \quad \forall x \in [0, 1]$$

Boundedness of Derivative is not necessary for a function to be of bounded Variation#

Boundedness of f' is not necessary for f to be of bounded variation. e.g

Let $f(x) = x^{1/3}$.

This function is monotone because

$$f'(x) = \frac{1}{3x^{2/3}} > 0 \quad \forall x \in \text{some finite interval}$$

Hence f is of bounded variation on any finite interval. However $f'(x) \longrightarrow \infty$ as $x \longrightarrow 0$ i.e $f'(x)$ is not bounded on finite interval containing zero.

The Sum $\sum_{i=1}^{n} |\Delta f_i|$ Increases on Refinement#

The sum $\sum_{i=1}^{n} |\Delta f_i|$ increases on the refinement of partition.

Consider a trivial partition $\{a, b\}$ of $[a, b]$ and let $\{a, c, b\}$ be its refinement. Then.

$$|f(b) - f(a)| \leq |f(b) - f(c)| + |f(c) - f(a)|$$

$\Rightarrow$ Thus the sum $\sum_{i=1}^{n} |\Delta f_i|$ increases on refinement i.e refinement has greater sum.

The sum, difference and Product of functions of bounded variations are of bounded Variation

Theorem# Let $f, g \in BV[a,b]$. Then prove that $f+g$, $f-g$ and fg are also of bounded variation on $[a,b]$. Also, we have $V_{f\pm g}(a,b) \leq V_f + V_g$

Proof #(a) Let f, g be functions of bounded variation on $[a,b]$ and let $P = \{x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a,b]$. Then

$$\sum_{i=1}^{n} |\Delta(f+g)_i| = \sum_{i=1}^{n} |(f+g)(x_i) - (f+g)(x_{i-1})|$$

$$= \sum_{i=1}^{n} |f(x_i) - f(x_{i-1}) + g(x_i) - g(x_{i-1})|$$

$$\leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$

$$\leq V(f; a,b) + V(g; a,b) \longrightarrow (1)$$

$\Rightarrow (f+g)(x)$ is of bounded variation on $[a,b]$

Also taking the supremum of the left-hand side in (1)

$$V(f+g; a,b) \leq V(f, a,b) + V(g; a,b)$$

Difference of functions

$$\sum_{i=1}^{n} |\Delta(f-g)_i| = \sum_{i=1}^{n} |[f(x_i) - f(x_{i-1})] + [g(x_i) - g(x_{i-1})]|$$

$$\leq \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$

$$\leq V(f;a,b)+V(g;a,b) < \infty \rightarrow ②$$

$\Rightarrow (f-g)$ is of bounded variation on $[a,b]$

Also taking supremum of left hand side of ②, we have

$$V(f-g;a,b) \leq V(f;a,b)+V(g;a,b)$$

<u>Note</u># Note that $V(-f;a,b) = V(f;a,b)$

<u>Product function</u>#

Let $h(x) = (fg)(x) = f(x)g(x)$.

$\because f,g \in BV[a,b]$

$\therefore f,g$ are bounded on $[a,b]$ and $\exists$ two numbers M_1, M_2 such that

$$|f(x)| \leq M_1 \quad \& \quad |g(x)| \leq M_2.$$

Let $P = \{a = x_0, x_1, x_2 \dots x_n = b\}$ be any partition of $[a,b]$

$$\sum_{i=1}^{n} |\Delta h_i| = \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})|$$

$$= \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_i)g(x_{i-1}) + f(x_i)g(x_{i-1}) - f(x_{i-1})g(x_{i-1})|$$

(adding and subtracting $f(x_i)g(x_{i-1})$)

$$= \sum_{i=1}^{n} |f(x_i)\{g(x_i) - g(x_{i-1})\} + g(x_{i-1})\{f(x_i) - f(x_{i-1})\}|$$

$$= \sum_{i=1}^{n} |f(x_i)\Delta g_i + g(x_{i-1})\Delta f_i|$$

$$\leq \sum_{i=1}^{n} |f(x_i)||\Delta g_i| + \sum_{i=1}^{n} |g(x_{i-1})||\Delta f_i|$$

$$\leq M_2 V(g;a,b) + M_1 V(f;a,b) \rightarrow ③$$

$\Rightarrow h = fg$ is of bounded variation on $[a,b]$
Also by taking supremum on L.H.S of ③, we have

$$V(h; a,b) \leq M_2 V(g; a,b) + M_1 V(f; a,b)$$

BV[a,b] is closed under multiplication by a Const:

Theorem# If $f \in BV[a,b]$ & $c \in R$, then $cf \in BV[a,b]$ and

$$V(cf; a,b) = |c| V(f; a,b)$$

Proof# Let $P = \{a = x_0, x_1, x_2, \dots x_n = b\}$ be any partition of $[a,b]$

$$\sum_{i=1}^{n} |\Delta (cf)_i| = \sum_{i=1}^{n} |(cf)(x_i) - (cf)(x_{i-1})|$$

$$= \sum_{i=1}^{n} |c f(x_i) - c f(x_{i-1})|$$

$$= |c| \sum_{i=1}^{n} | f(x_i) - f(x_{i-1})|$$

$$= |c| \sum |\Delta f_i| \leq |c| V(f, a:b) < \infty$$

$\Rightarrow cf \in BV[a,b]$
Also taking supremum on L.H.S. we get

$$V(cf; a,b) \leq |c| V(f; a,b)$$

Remarks# By above results we note that $BV[a,b]$ is a commutative ring with identity $f=1$ (a function with total variation zero). Now it is natural to ask for the units in $BV[a,b]$ i.e for function in $BV[a,b]$ for which reciprocal is also in $BV[a,b]$. Now the reciprocal of a function of bounded variation need not

to be a function of bounded variation. If f vanishes at some point of $[a,b]$, then $\frac{1}{f}$ is not even defined on all of $[a,b]$. But since f may not be continuous, therefore it is possible that f not vanish at any point of $[a,b]$, yet $1/f$ not be in $BV[a,b]$ e.g if $\lim_{x\to c} f(x)=0$, then the function $1/f$ is unbounded even though $f(c)\neq 0$. Since every function of bounded variation is already bounded, therefore $1/f$ can not be in $BV[a,b]$.
However if f is of bounded variation and is bounded away from 0 on all of $[a,b]$, then $1/f$ is also in $BV[a,b]$.
We prove this in the following theorem

The reciprocal of a function of bounded variation which is bounded away from zero, is of bounded Variation

Theorem # If a function f is of bounded variation on $[a,b]$ and is bounded away from zero on $[a,b]$, then $1/f$ is also of bounded variation on $[a,b]$

OR

Suppose that f is of bounded variation on $[a,b]$ and there exists a number m such that $|f(x)|\geq m>0$ i.e f is bounded away from zero $\forall x\in[a,b]$. Then $1/f$ is also of bounded variation on $[a,b]$ and

$$V\left(\frac{1}{f};a,b\right)\leq \frac{V(f;a,b)}{m^2}$$

Proof # Let $g=1/f$. Let $P=\{a=x_0,x_1,x_2,\dots x_n=b\}$ be any partition of $[a,b]$.

$$\sum_{i=1}^{n}|\Delta g_i| = \sum_{i=1}^{n}\left|\frac{1}{f(x_i)}-\frac{1}{f(x_{i-1})}\right| = \sum\frac{|f(x_{i-1})-f(x_i)|}{|f(x_i)||f(x_{i-1})|}$$

$$\leq \frac{1}{m^2} |\Delta f_i| \leq \frac{1}{m^2} V(f; a,b) < \infty$$

$\Rightarrow g = 1/f$ is of bounded variation on $[a,b]$.
Also by taking supremum of the L.H.S. we have

$$V(\tfrac{1}{f}, a,b) \leq \frac{1}{m^2} V(f, a,b)$$

<u>Example</u> # Any polynomial p is in $BV[a,b]$, for any interval $[a,b]$. The polynomial p is a unit in $BV[a,b]$ i.e $\frac{1}{p} \in BV[a,b]$ for any interval containing none of its zeros.

<u>Example</u> # The exponential function e^x is in $BV[a,b]$ for any interval $[a,b]$. More generally, the function $e^{u(x)}$ is of bounded variation for interval $[a,b]$ on which $u(x)$ has a bounded derivative or on which $u(x)$ is monotone.

<u>Example</u> # The trignometric functions $\sin x$ and $\cos x$, having bounded derivatives are in $BV[a,b]$ for any interval $[a,b]$. Consequently by above theorem $\sec x = \frac{1}{\cos x}$ is in $BV[a,b]$ for any intervals $[a,b]$ containing none of the points $\{(2n+1)\pi/2 : n \in \mathbb{Z}\}$.
Also since product of two functions of bounded variation is of bounded variation, therefore $\tan x = \frac{\sin x}{\cos x}$ is of bounded variation on any interval $[a,b]$ containing none of the points $\{(2n+1)\pi/2 : n \in \mathbb{Z}\}$ i.e zeros of $\cos x$.

<u>Example</u> #
Any rational function p/q is in $BV[a,b]$ for any interval $[a,b]$ containing none of zeros of $q(x)$.

Total Variation as a Function # OR
Variation function of a function of bounded variation

Upto now we have kept the interval $[a,b]$ fixed and $V(f;a,b)$ was considered as a function of f. If we keep the function f fixed and allow the interval under consideration to vary we can study the total variation of f as a function of the interval and can define the variation function of f over $[a,b]$. Before defining the variation function of a function of bounded variation we first prove that total variation V_f is additive on intervals and on the basis of this property we define total variation as function of interval.

Total Variation is additive on intervals

Theorem #. Let f be in $BV[a,b]$. Let c be any point in (a,b). Then f is in $BV[a,c]$ and in $BV[c,b]$ and conversely. Also

$$V_f(a,b) = V_f(a,c) + V_f(c,b)$$

OR

If $f \in BV[a,b]$, then f is of bounded variation on any closed sub-interval of $[a,b]$ and if $c \in (a,b)$, we have

$$V_f(a,b) = V_f(a,c) + V_f(c,b)$$

Proof # Let f be of bounded variation on $[a,b]$. We prove that f is of bounded variation on $[a,c]$ and $[c,b]$.

Let $P_1 = \{a = x_0, x_1, x_2 - - - x_m = c\}$

and $P_2 = \{y_0 = c, y_1, y_2 - - - , y_n = b\}$ be partitions

of $[a,c]$ and $[c\ b]$ respectively. Then

$$P' = P_1 \cup P_2 = \{a = x_0, x_1, x_2 \ldots\ldots x_m, y_0, y_1 \ldots\ldots y_n\}.$$ is a partition

of $[a,b]$ and we have

$$\sum_{i=1}^{n} |\Delta f_i| \leq V(f; a,b)$$

$$\Rightarrow \sum_{i=1}^{m} |\Delta f_i| + \sum_{j=1}^{n} \Delta f_j \leq V(f, a,b)$$

$$\Rightarrow \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| + \sum_{j=1}^{n} |f(y_j) - f(y_{j-1})| \leq V(f; a,b) \rightarrow ①$$

$$\Rightarrow \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})| \leq V(f; a,b) \rightarrow ②$$

and $$\sum_{j=1}^{n} |f(y_j) - f(y_{j-1})| \leq V(f; a,b) \rightarrow ③$$

From ② and ③ we note that for any partitions P_1, P_2 of $[a,c]$ & $[c,b]$ $\sum_{i=1}^{m} |\Delta f_i|$ and $\sum_{j=1}^{n} |\Delta f_j|$ are bounded by $V(f; a,b)$

$$\Rightarrow f \in BV[a,c] \text{ and } f \in BV[c,b]$$

From ① taking supremum when P_1 varies in $\mathcal{P}[a,c]$, we have

$$V(f; a,c) + \sum_{j=1}^{n} |f(y_j) - f(y_{j-1})| \leq V(f; a,b)$$

Again taking supremum when P_2 varies in $\mathcal{P}[c,b]$ we have

$$V(f; a,c) + V(f; c,b) \leq V(f; a,b) \rightarrow ⓐ$$

Converse # Conversely suppose that f is of bounded variations on $[a,c]$ & $[c,b]$. We prove that $f \in BV[a,b]$.

Let $P = \{a = x_0, x_1, x_2, x_3 - - - x_p = b\}$ be any partition of $[a,b]$. We may assume without loss of generality that c is a point division of P. Otherwise adjoin c to P to obtain a refinement of P. Suppose that $c = x_k$ and $k < p$. Then P is union of partitions

$$P_1' = \{x_0, x_1, x_2, \dots x_{k-1}, x_k = c\} \text{ of } [a,c]$$

and

$$P_2' = \{c = x_k, x_{k+1}, \dots x_p\}.$$

Then

$$\sum_{i=1}^{p} |\Delta f_i| = \sum_{i=1}^{k} |\Delta f_i| + \sum_{i=k+1}^{p} |\Delta f_i|$$

$$\leq V(f; a,c) + V(f; c,b) \rightarrow (4)$$

$\Rightarrow$ f is of bounded variation on $[a,b]$.

Also taking supremum of L.H.S of (4) as P varies over $\mathcal{P}[a,b]$, we get

$$V(f; a; b) \leq V(f; a; c) + V(f; c, b) \rightarrow (b)$$

By (a) & (b) we have

$$V(f; a:b) = V(f; a:c) + V(f: c, b) \text{ Proved.}$$

Remarks # This theorem enables us to create, for any f in $BV[a,b]$, a new function that measures variation of f over the interval $[a,x]$

Example # As an application of above Theorem, we compute the total variation of

$$f(x) = |x| \text{ on } [-3, 1].$$

Since f is increasing on $[0, 1]$,

$$V(f; 0, 1) = f(1) - f(0) = 1 - 0 = 1$$

Similarly

$$V(f; -3, 0) = f(0) - f(-3) = 3$$

Thus

$$V_{-3}^{1} f = V_{-3}^{0} f + V_{0}^{1} f$$

$$= 1 + 3 = 4$$

Variation Function (Definition)

Let f be a function of bounded variation over an interval $[a, b]$. If x is any point of $[a, b]$, then a function $V(x)$ defined by

$$V_f(x) = V(x) = V(f : a : x) \quad \forall x \in (a, b]$$

, where $V(f : a : a) = 0$

is called variation function of f over $[a, b]$.

Monotonically Increasing Character of the Variation Function

Theorem # Let f be of bounded variation on $[a, b]$ and $V_f(x)$ be variation function of f over $[a, b]$. Then

(a) # $V_f(x)$ is an increasing function of x on $[a, b]$

(b) # $V_f(x) - f(x)$ is an increasing function on $[a, b]$

Proof# (a)# Let x, y be two points of $[a, b]$ such that

$$a < x < y \leq b.$$

Then

$$V_f(a:y) = V_f(a,x) + V_f(x,y)$$

$$\Rightarrow V(y) = V(x) + V_f(x,y)$$

$$\Rightarrow V(y) - V(x) = V_f(x,y) \geq 0$$

$$\Rightarrow V(y) - V(x) \geq 0$$

$$\Rightarrow V(y) \geq V(x)$$

$\because$ x, y are any two arbitrary points of $[a.b]$

$\therefore$ $V(x)$ is increasing on $[a, b]$.

(b)# Let $x, y \in [a, b]$ such that

$$a \leq x < y \leq b.$$

Let $D(x) = V(x) - f(x)$

Then

$$D(y) - D(x) = V(y) - f(y) - [V(x) - f(x)]$$

$$= V(y) - V(x) + f(x) - f(y)$$

$$= V(f: x, y) - [f(y) - f(x)] \to ①$$

Now the total variation of f on $[x, y]$ is

$$V(f; x, y) = \sup\{\sum_{i=1}^{n} |\Delta f_i| : P \text{ in } \mathcal{P}[x, y]\}$$

Since $\{x, y\}$ is trivial partition of $[x, y]$, therefore

$$f(y) - f(x) \leq V(f; x, y)$$

and from ①, we have

$$D(y) - D(x) = V(f; x, y) - [f(y) - f(x)] \geq 0$$

$\Rightarrow D = V_f - f$ is an increasing on $[a,b]$

Note # The above theorem enable us to prove the following important and elegant charcterisation of functions in $BV(a,b)$

Characterisation of Functions of Bounded Variation

Theorem # # Let f be defined on $[a,b]$. Then f is of bounded variation on $[a.b]$ iff it can be expressed as the difference of two monotone increasing (monotone non-decreasing) functions.

OR

$BV(a,b)$ Consists exactly those functions f that can be written as the difference of two monotone increasing functions

Proof # Let f be of bounded variation on $[a,b]$ and $V(f;x) = V_f(x)$ be its variation function on $[a,b]$. Then

$$f(x) = \frac{1}{2}[V(x) + f(x)] - \frac{1}{2}[V(x) - f(x)]$$

$$= g(x) + h(x) \quad \text{say}$$

We show that $g(x)$ & $h(x)$ are monotonically increasing on $[a,b]$.

if $x_2 > x_1$, $x_1, x_2 \in [a,b]$, then

$$g(x_2) - g(x_1) = \frac{1}{2}[V(x_2) + f(x_2)] - \frac{1}{2}[V(x_1) + f(x_1)]$$

$$= \frac{1}{2}[V(x_2) - V(x_1)] - \frac{1}{2}[f(x_1) - f(x_2)]$$

$$= \frac{1}{2}\{V_f(x_1, x_2) - \{f(x_1) - f(x_2)\}]$$

$\because$ $V_f(x_1, x_2) \geq |f(x_1) - f(x_2)|$

$\therefore$ $V_f(x_1, x_2) \geq f(x_1) - f(x_2)$

$\Rightarrow$ $g(x_2) - g(x_1) \geq 0$

$\Rightarrow$ $g(x_2) \geq g(x_1)$

$\Rightarrow$ g is monotonically increasing on $[a, b]$

Again we have

$$h(x_2) - h(x_1) = \frac{1}{2}[\{V(x_2) - V(x_1)\} - \{f(x_2) - f(x_1)\}]$$

$$= \frac{1}{2}[V(f; x_1, x_2) - \{f(x_2) - f(x_1)\}]$$

and as before

$h(x_2) - h(x_1) \geq 0$

$\Rightarrow$ $h(x_2) \geq h(x_1)$

$\Rightarrow$ h is monotonically increasing on $[a, b]$.

Hence f is expressed as the difference of two monotonically

Converse# Conversely suppose that

$$f(x) = f_1(x) - f_2(x)$$

where f_1 & f_2 are monotonically increasing functions on $[a, b]$. We are to prove that f is of bounded variation on $[a, b]$

$\because$ A monotone function is of bounded variation
$\therefore$ f_1, f_2 being monotone are both of bounded variation on $[a, b]$

Again since the difference of two functions of bounded variation is of bounded variation, Therefore $f_1 - f_2$ is of bounded variation

Hence $f = f_1 - f_2$ is also of bounded variation

(Proved)

Method # II

The above Theorem can also be proved as under.

Let f be of bounded variation and $V(x)$ be variation function of f on $[a, b]$. defined by

$$V(x) = \begin{cases} V(f : a : x) & a < x \leq b \\ 0 & x = a \end{cases}$$

Then $V(x)$ is an increasing function which can be proved as under.

Let $x, y \in [a, b]$ such that $x < y$

Then

$$V(f; a, y) = V(f : a, x) + V(f; x, y)$$

$$\Rightarrow V(f; a, y) - V(f : a, x) = V(f; x, y) \geq 0$$

$$\Rightarrow V(y) \geq V(x)$$

$\Rightarrow$ V is monotonically increasing

Again function $g(x) = V(x) - f(x)$ is monotonically increasing on $[a, b]$ because if $a < x < y \leq b$, then

* The representation of a function of bounded variation as a difference of two increasing functions is not unique because if

$$f = f_1 - f_2$$

where f_1 & f_2 ↑ on $[a,b]$. Then for any increasing function g on $[a,b]$, we have

$$f = (f_1+g) - (f_2+g)$$

a 2nd representation of f as the difference of two ↑ functions on $[a,b]$ because f_1+g, f_2+g are ↑ on $[a,b]$. If g is strictly increasing, then same is true for f_1+g and f_2+g. Therefore the above theorem also holds iff g is strictly increasing function.

Example # Let $f(x) = \cos x$ on $[0, 2\pi]$.

Let $g(x) = \begin{cases} 0 & \text{for } 0 \le x \le \pi \\ 1+\cos x & \text{for } \pi < x \le 2\pi \end{cases}$

and

$h(x) = \begin{cases} -\cos x & 0 \le x \le \pi \\ 1 & \pi < x \le 2\pi \end{cases}$

Then both g and h are monotone increasing on $[0, 2\pi]$ and.

$$f = g - h.$$

Problem # Drichlet's dancing function on $[0,1]$ is defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ is ration} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Prove that f is of bounded variation on $[0,1]$

$$g(y) - g(x) = V(y) - f(y) - [V(x) - f(x)]$$

$$= V(y) - V(x) - [-f(x) + f(y)]$$

$$= V(f; x; y) - [f(y) - f(x)]$$

But $V(f; x; y) \geq f(y) - f(x)$

$\Rightarrow g(y) - g(x) \geq 0$

$\Rightarrow g(y) \geq g(x)$

$\Rightarrow g$ is $\uparrow$ on $[a, b]$

Now $g(x) = V(x) - f(x)$

$\Rightarrow f(x) = V(x) - g(x)$

$\Rightarrow f$ is expressed as the difference of two monotonically increasing functions on $[a, b]$

Converse # Conversely let

$$f(x) = f_1(x) - f_2(x)$$

where f_1 & f_2 are monotonically increasing function on $[a, b]$. Then

f_1, f_2 being monotone are of bounded variation on $[a, b]$ and their difference is also of bounded variation on $[a, b]$.

Hence f is of bounded variation on $[a, b]$

Remarks # ★ The above theorem shows that the the monotonically increasing functions are in a sense the only functions of bounded variation in a sense that every function of bounded variation can be written as the difference of two monotonically increasing functions.

Sol # Consider a partition $P=\{x_0<x_1<x_2 \dots x_n=1\}$ of $[0\ 1]$ such that 1st end points of partition intervals are irrational and 2nd end points are rational. Then for this P, we have

$$\sum_{i=1}^{n}|\Delta f_i| = \sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$$

$$= |f(x_1)-f(x_0)| + |f(x_2)-f(x_1)| + \dots + |f(x_n)-f(x_{n-1})|$$

$$= |0-1| + |0-1| + |0-1| + \dots + |0-1|$$

$$= 1+1+1+1+\dots + 1 \quad (n \text{ terms})$$

$$= n$$

which will not be bounded because $\sum n$ is divergent and the sequence of its partial sum $\left\{\sum_{i=1}^{n}|\Delta f_i|\right\}_{n=1}^{\infty}$ is unbounded

Variation function of Continuous functions of Bounded Variation

Theorem # Let f be a function of bounded variation on $[a.b]$. Then $f(x^+)=\lim_{n\to x^+}f(x)$ exists at every point x in $[a,b)$ and $f(x^-)$ exists at every point point x in $(a,b]$

Proof # Let $V_f(x)$ be variation function of f. Then V_f and V_f-f are monotone increasing on $[a,b]$ and therefore $V_f(x^+)$, $(V_f-f)(x^+)$ exist at every point x in $[a,b)$. Since

$$f(x) = V_f(x) - (V_f(x) - f(x))$$

Therefore it follows that

$$f(x^+) = V_f(x^+) - [V_f(x^+) - f(x^+)].$$

exists at every x in $[a,b)$. The left hand limits are treated similarly.

Theorem# Let f be of bounded variation on $[a,b]$. Then f is continuous at a point c in $[a,b]$ iff variation function $V_f(x)$ is continuous at c i.e a point of continuity of f is also a point of continuity of V and conversely

Proof# Suppose that $V(x)$ is continuous at point c.

Then for given $\epsilon > 0$ $\exists$ a $\delta > 0$ such that

$$|V(x) - V(c)| < \epsilon \quad \text{for } |x-c| < \delta \rightarrow ①$$

Also we have

$$|f(x) - f(c)| \leq V(x) - V(c) \quad \text{if } x > c \rightarrow ②$$

and $$|f(x) - f(c)| \leq V(c) - V(x) \quad \text{if } x < c \rightarrow ③$$

from ① & ② we have

$$\boxed{\cancel{-\{V(x) - V(c)\} \leq |f(x) - f(c)| \leq V(x) - V(c)}}$$

$$|f(x) - f(c)| \leq V(x) - V(c) \quad \text{if } x > c \text{ \& } x - c < \delta \rightarrow ④$$

i.e if $x \in (c, c+\delta)$

and from ③ we deduce that

$$-\{V(x) - V(c)\} \leq |f(x) - f(c)| \quad \text{if } x < c \text{ \& if } c - \delta < x \text{ or } x \in (c-\delta, c) \rightarrow ⑤$$

from ④ & ⑤, we have

$$\left||f(x)-f(c)|\right| \leq V(x)-V(c) \quad \forall x \text{ s.t } |x-c|<\delta \rightarrow ⑦$$

By ① & ⑥

$$|f(x)-f(c)| < \epsilon \quad \forall x \text{ s.t. } |x-c|<\delta$$

$\Rightarrow f$ is Continuous at $x=c$

Converse#

Conversely suppose that c is a point of Continuity of f. We prove that $V(x)$ is continuous at $x=c$.

$\because f$ is continuous at $x=c$

$\therefore$ for $\epsilon>0$ $\exists$ a $\delta>0$ such that

$$|f(x)-f(c)| < \frac{\epsilon}{2} \quad \text{for } |x-c|<\delta \rightarrow ②$$

Since $V(f;c:b) = \text{Sup}\{\sum_{i=1}^{n}|\Delta f_i| : P \in \mathcal{P}[a,b]\}$

Therefore by property of Supremum for the same $\epsilon>0$ there exists a partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[c,b]$ such that

$$V(f;c,b)-\epsilon/2 < \sum_{i=1}^{n}|\Delta f_i| \leq V(f;c,b)$$

$$V(f;c,b)-\epsilon/2 < \sum_{i=1}^{n}|\Delta f_i| \leq V(f;c,b) \rightarrow ①$$

Refining the partition P only increases the sum $\sum_{i=1}^{n}|\Delta f_i|$ i.e. inequality in ① remains valid. Hence we can assume that P is sufficiently refined so that

guage $\|P\|$ of P is less than δ. This implies in particular that partition point x_1 of P satisfies the relation.

$0 < |x_1 - c| < \delta$.

Therefore according to ⓐ, we have

$$|f(x_1) - f(c)| < \epsilon/2$$

Isolate the 1st summand in the inequality ① and note that $\{x_1, x_2, x_3 \dots x_n\}$ is a partition of $[x_1, b]$.

$$V(f; c, b) - \epsilon/2 < |f(x_1) - f(c)| + \sum_{i=2}^{n} |\Delta f_i|$$

$$\leq \epsilon/2 + V(f; x_1, b)$$

$$\Rightarrow V(f; c, b) - V(f; x_1, b) < \epsilon/2 + \epsilon/2 = \epsilon$$

$$\Rightarrow V(f; c, b) - V(f; x_1, b) < \epsilon \longrightarrow ②$$

But

$$0 \leq V_f(x_1) - V_f(c) = V(f; a, x_1) - V(f; a, c)$$
$$= V(f; c, x_1)$$
$$= V(f; c, b) - V(f; x_1, b) < \epsilon$$

$$\Rightarrow |V_f(x_1) - V_f(c)| < \epsilon \qquad 0 < |x_1 - c| < \delta$$

$\because x_1 > c$ & $x_1 - c < \delta \Rightarrow x_1 \in (c, c+\delta)$.

Thus we have proved that

$$|V_f(x) - V_f(c)| < \epsilon \qquad \text{whenever } x \in (c, c+\delta)$$

$$\Rightarrow V_f(c^+) = V_f(c) \longrightarrow ③$$

$\Rightarrow V_f(x)$ is continuous from the right at c.

Now we choose a partition $P = \{y_0, y_1, \dots y_m\}$ of $[a, c]$ such that $\|P\| < \delta$

and For this partition

$$V(f;a.c)-\epsilon/2 < \sum_{i=1}^{m}|\Delta f_i| \rightarrow (4)$$

Again note that $\{y_0, y_1, \dots, y_{m-1}\}$ is a partition of $[a, y_{m-1}]$ and since $0<|y_{m-1}-c|<\delta$, therefore

$$|f(c)-f(y_{m-1})| < \epsilon/2 \rightarrow (5) \text{ by } (a)$$

from (4)

$$V_f(c)-\epsilon/2 = V(f;a;c)-\epsilon/2 < \sum_{i=1}^{m}|\Delta f_i|$$

$$= \sum_{i=1}^{m-1}|\Delta f_i| + |f(c)-f(y_{m-1})|$$

$$< V(f;a,y_{m-1})+\epsilon/2 \quad \text{by } (5)$$

$$\Rightarrow V(f;a,c)-V(f;a,y_{m-1}) < \epsilon \longrightarrow (6)$$

Now

$$0 \le V_f(c)-V_f(y_{m-1}) = V(f;a,c)-V(f;a;y_{m-1})$$

$$= V(f;y_{m-1},c) < \epsilon \quad \text{by } (6)$$

$$\Rightarrow |V_f(c)-V_f(y_{m-1})| < \epsilon \quad \text{when } |y_{m-1}-c|<\delta$$

$\because$ $c > y_{m-1}$ & $c-y_{m-1} < \delta \Rightarrow y_{m-1} \in (c-\delta, c)$

$$\Rightarrow V_f(c^-) = V_f(c) \rightarrow (7)$$

$\Rightarrow$ V_f is continuous from left at c. Therefore V_f is continuous at c. Hence theorem is proved.

Note # Theorem also holds if "increasing" is replaced by strictly increasing.

Theorem# Let f be continuous on $[a,b]$. Then f is of bounded variation on $[a,b]$ iff f can be expressed as the difference of two monotone increasing continuous functions.

OR

A continuous function is of bounded variation iff it can be expressed as the difference of two continuous monotonically increasing functions.

Proof# Let f be continuous and of bounded variation on $[a,b]$. Let $V_f(x)$ the variation function of f, on $[a,b]$. Then by above Theorem every point of continuity of f is also a point of continuity of $V_f(x)$. Since f is continuous on $[a,b]$, therefore $V_f(x)$ is also continuous on $[a,b]$.

Also $V_f(x)$ is monotonically increasing and $V(x)-f(x)$ is monotonically increasing.

$\Rightarrow$ $V_f(x)+(-f(x))$ being sum of two continuous function is continuous monotonically increasing function.

Also

$$f(x) = V_f(x) - [V_f(x) - f(x)]$$

$\Rightarrow$ f is expressed as the difference of two continuous monotonically increasing function.

Conversely let $f = f_1 - f_2$, where f_1, f_2 are continuous and $\uparrow$ on $[a,b]$. Then f_1, f_2 being $\uparrow$ on $[a,b]$ are of bounded variation on $[a,b]$ and hence f is also of bounded variation on $[a,b]$ because difference of two functions of bounded variation is a function of bounded variation.

تمت بالخیر